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VOLUME XIV NUMBER 3

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED BY

EDWARD L. NICHOLS

ERNEST MERRITT, and FREDERICK BEDELL

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MARCH 1902

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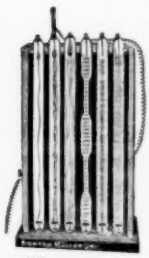

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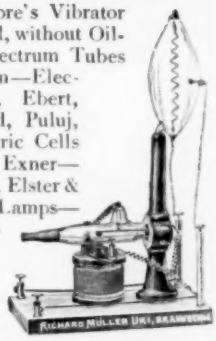

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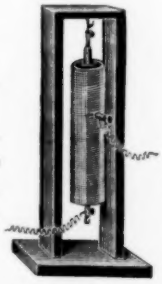

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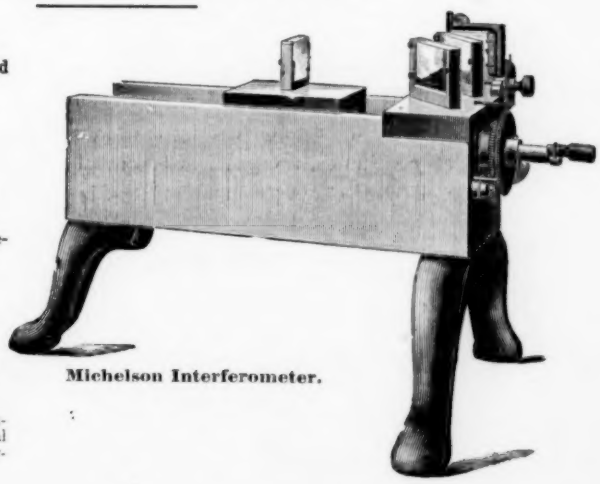

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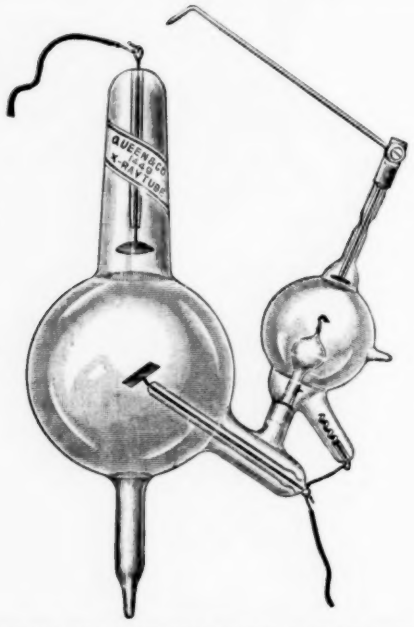

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Volume XIV. *March, 1902.* *Number 3.*

THE

PHYSICAL REVIEW.

THE ELASTIC PROPERTIES OF HELICAL SPRINGS.

By Joseph Warren Miller, Jr.

1. *Historical Note.*—In 1814 M. J. Binet[1] noticed the fundamental principle that helical springs act chiefly by torsion. Apparently nothing was done in this line between that time and 1848, when James Thomson[2] showed that a helical spring, of infinitely small slope, acts exactly as a torsion balance using the same wire straightened. He verified his theory by experiments. The fact that a spring acts as a torsion balance means it follows Hooke's law. It is this close accord with Hooke's law that is utilized in the ordinary spring balance. Apparently a complete statement of the theory of helical springs was first given by Kelvin and Tait in their "Natural Philosophy,"[3] 1867. Certain aspects of the theory were also given by Clebsch[4] and Kirchhoff.[5] Some of the most recent contributions upon this subject are those of Prof. A. E. H. Love, who devotes several pages of his "Theory of Elasticity,"[6] to helical springs, and who also published a paper on "The Propagation of Waves of Elastic Displacement along a Helical Wire," which appeared in the volume of "Memoirs on the occasion of the jubilee of Sir George G. Stokes."[7]

[1] Journal de L'École Royal Polytechnique, Tome X., p. 419.
[2] Camb. and Dub. Math. Journal, Vol. III., p. 258.
[3] Vol. II., Arts. 604–607.
[4] Vorlesungen über Mathematische Physik, Mechanik, 1876.
[5] Theorie der Elasticität fester Körper, 1864.
[6] Vol. II., Chapters XIV.–XVI., 1892.
[7] p. 346, 1900.

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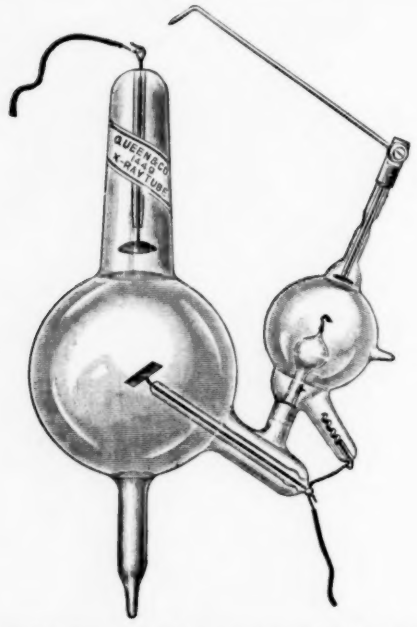

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The purpose of the present paper is to set forth the results of a research conducted in the Department of Mechanics in Columbia University. The original object of this enquiry was to learn whether helical springs could be used to determine the acceleration due to gravity. Early experiments showed the need of a more extended enquiry into the behavior of springs under static conditions. Thus the following investigation refers chiefly to the static constants and properties of a set of helical springs of different dimensions and metals.

2. *General Theory.*—To define the bending and twisting essential in a wire whose elastic central line assumes the form of a helix of slope α on a cylinder of radius r, consider the relations shown in the accompanying diagram.

At any point O on the elastic central line draw in the plane tangent to the cylinder the tangent to the helix OX', the element OZ, the horizontal line OX, and the line OZ' perpendicular to OX'. Let ds be an element of the elastic central line measured from O towards X'. The angular rotation about the axis of the cylinder corresponding to this element, or what is the same thing, the angle between two meridian planes through the ends of ds, is $ds \cos \alpha/r$. This rotation is equivalent to its components about OZ' and OX' that is to $ds \cos \alpha \cos \alpha/r$ and $ds \cos \alpha \sin \alpha/r$. The first of these is a pure bending and the second is a pure twisting when applied to the wire. Thus for a right cylindrical helix the amounts of bending and twisting per unit length of the wire are $\cos^2 \alpha/r$ and $\cos \alpha \sin \alpha/r$ respectively.[1] The potential energy of a wire of length s, measured along its elastic central line, bent into a helix of slope α and radius r is hence easily deduced. For the part due to bending is proportional to $(\cos^2 \alpha/r)^2$ per unit length, and the part due to twisting is proportional to $(\cos \alpha \sin \alpha/r)^2$ per unit length. Therefore the work done in producing the helix is

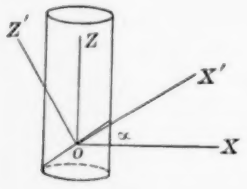

Fig. 1.

$$\tfrac{1}{2} s \left\{ A \frac{\cos^2 \alpha \sin^2 \alpha}{r^2} + B \frac{\cos^4 \alpha}{r^2} \right\},$$

[1] Compare Thompson and Tait's Natural Philosophy, Part II., Art. 602, p. 137.

where A is the nodulus of twist, and B is the nodulus of bending.

If we write

$$\begin{aligned} z &= s \sin \alpha, & \varphi r &= s \cos \alpha, \\ z_0 &= s \sin \alpha_0, & \varphi_0 r_0 &= s \cos \alpha_0; \end{aligned} \tag{1}$$

where z_0 is the axial length of the helix, r_0 is the radius of the cylinder containing the elastic central line of the wire, and φ_0 the angle between two planes through the ends of the helix and its axis in the unstrained condition; and, similarly, z, r, and φ the corresponding quantities for the strained condition, then the curvature

$$\cos^2 \alpha / r = \varphi \cos \alpha / s$$

and the change in curvature, per unit length, is

$$(\varphi \cos \alpha - \varphi_0 \cos \alpha_0)/s.$$

The potential energy per unit length, due to the bending, is proportional to $(\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2/s^2$, and hence this part of the energy for the whole wire is

$$\tfrac{1}{2} \frac{B}{s} (\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2.$$

Similarly the change in twist per unit length is

$$\frac{\cos \alpha \sin \alpha}{r} - \frac{\cos \alpha_0 \sin \alpha_0}{r_0} = (\varphi \sin \alpha - \varphi_0 \sin \alpha_0)/s;$$

whence the potential energy due to twist alone is

$$\tfrac{1}{2} \frac{A}{s} (\varphi \sin \alpha - \varphi_0 \sin \alpha_0)^2.$$

Calling the entire potential energy V, it follows that

$$V = \tfrac{1}{2} \frac{A}{s} (\varphi \sin \alpha - \varphi_0 \sin \alpha_0)^2 + \tfrac{1}{2} \frac{B}{s} (\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2. \tag{2}$$

Let

P = longitudinal pull

and

L = torsional moment.

Then we have

$$P = \frac{dV}{dz} = \frac{dV}{da}\frac{da}{dz} = \frac{1}{s\cos a}\frac{dV}{da},$$

or

$$P = \frac{A}{s^2}(\varphi \sin a - \varphi_0 \sin a_0)\varphi + \frac{B}{s^2}(\varphi_0 \cos a_0 - \varphi \cos a)\varphi \tan a, \quad (3)$$

and

$$L = \frac{dV}{d\varphi} = \frac{A}{s}(\varphi \sin a - \varphi_0 \sin a_0)\sin a + \frac{B}{s}(\varphi \cos a - \varphi_0 \cos a_0)\cos a. \quad (4)$$

If the applied couple L is zero equation (4) gives

$$\frac{A}{B} = \frac{\cos a}{\sin a}\cdot\frac{\varphi_0 \cos a_0 - \varphi \cos a}{\varphi \sin a - \varphi_0 \sin a_0}. \quad (5)$$

If we introduce (5) in (3) there results

$$\frac{Ps^2}{A} = (\varphi \sin a - \varphi_0 \sin a_0)\varphi \sec^2 a. \quad (6)$$

Expanding the second member of this by Maclaurin's series and solving for P we get

$$P = \frac{A\varphi_0^2}{s^2}\sec^2 a\left\{\sin a - \sin a_0 + (2\sin a - \sin a_0)\frac{\Delta\varphi}{\varphi_0} + \sin a\left(\frac{\Delta\varphi}{\varphi_0}\right)^2\right\}, \quad (7)$$

$$= \frac{A\varphi_0^2}{s^3}\sec^2 a\left\{z - z_0 + (2z - z_0)\frac{\Delta\varphi}{\varphi_0} + z\left(\frac{\Delta\varphi}{\varphi_0}\right)^2\right\}. \quad (8)$$

From (8) we see that P varies nearly as $(z - z_0)\sec^2 a$, since $\frac{\Delta\varphi}{\varphi_0}$ is generally small. If we divide (3) by B and then introduce (5) we get

$$\frac{Ps^2}{B} = (\varphi_0 \cos a_0 - \varphi \cos a)\varphi(\tan a + \cot a), \quad (9)$$

$$= (\varphi_0 \cos a_0 - \varphi \cos a)2\varphi/\sin 2a.$$

As before, expanding the second member by Maclaurin's series,

$$P = \frac{2B\varphi_0^2}{s^2 \sin 2\alpha}\left\{\cos\alpha - \cos\alpha_0 + (\cos\alpha_0 - 2\cos\alpha)\frac{\Delta\varphi}{\varphi_0} - \cos\alpha\left(\frac{\Delta\varphi}{\varphi_0}\right)^2\right\}. \tag{10}$$

From (7) and (10) we have

$$\frac{1}{A} = \frac{\varphi_0^2 \sec^2\alpha}{Ps^2}\left\{\sin\alpha - \sin\alpha_0 + (2\sin\alpha - \sin\alpha_0)\frac{\Delta\varphi}{\varphi_0} + \sin\alpha\left(\frac{\Delta\varphi}{\varphi_0}\right)^2\right\}, \tag{11}$$

$$\frac{1}{B} = \frac{2\varphi_0^2}{Ps^2 \sin 2\alpha}\left\{\cos\alpha_0 - \cos\alpha - (2\cos\alpha - \cos\alpha_0)\frac{\Delta\varphi}{\varphi_0} - \cos\alpha\left(\frac{\Delta\varphi}{\varphi_0}\right)^2\right\}. \tag{12}$$

In case $\Delta\varphi$ is negligible these formulas give A and B independently. In general, however, terms in $\Delta\varphi$ may not be neglected in (12). For springs of small slope (11) will usually give a close approximation to A. Either formula may be used for exact computations when $\Delta\varphi$ or A/B is known, as shown below.

From (5) we get

$$\frac{\varphi}{\varphi_0} = \frac{B\cos\alpha_0 + A\sin\alpha_0\tan\alpha}{B\cos\alpha + A\sin\alpha\tan\alpha} = \frac{B\cos\alpha\cos\alpha_0 + A\sin\alpha\sin\alpha_0}{B\cos^2\alpha + A\sin^2\alpha} \tag{13}$$

$$= \frac{\cos(\alpha - \alpha_0) + x\cos(\alpha + \alpha_0)}{1 + x\cos 2\alpha},$$

wherein

$$x = \frac{B - A}{B + A}.$$

To compute the change in φ the following formula is best

$$\frac{\varphi}{\varphi_0} - 1 = \frac{\varphi - \varphi_0}{\varphi_0} = \frac{\Delta\varphi}{\varphi_0}$$
$$= 2 \sin \tfrac{1}{2}(\alpha - \alpha_0) \frac{x \sin \tfrac{1}{2}(3\alpha + \alpha_0) - \sin \tfrac{1}{2}(\alpha - \alpha_0)}{1 + x \cos 2\alpha}. \tag{14}$$

From (14) we see that $\Delta\varphi = 0$ for two values of α, namely, for $\alpha = \alpha_0$ and for the value obtained from

$$x \sin \tfrac{1}{2}(3\alpha + \alpha_0) = \sin \tfrac{1}{2}(\alpha - \alpha_0),$$

and, approximately, from

$$\alpha - \alpha_0 = x \sin 2\alpha_0 + \tfrac{3}{2} x^2 \sin 4\alpha_0 + \cdots.$$

From (14) we also see that as the spring elongates $\varphi - \varphi_0$ increases up to a maximum and then decreases. If we write $\xi = A/B$, then $x = (1 - \xi)/(1 + \xi)$, and the maximum of φ or $\Delta\varphi$ is given by the following cubic equation in $\tan \alpha$

$$\tan \alpha = \frac{\xi}{2\xi - 1} \tan \alpha_0 \frac{1 + (2 - \xi) \tan^2 \alpha}{1 + \dfrac{\xi}{2\xi - 1} \tan^2 \alpha}. \tag{15}$$

Since, for the case here considered, $\frac{2}{3} < \xi < 1$, as shown below by equation (21),

$$2 - \xi < \frac{\xi}{2\xi - 1},$$

a real root of the above cubic lies between

$$(2 - \xi) \tan \alpha_0 \quad \text{and} \quad \frac{\xi}{2\xi - 1} \tan \alpha.$$

From the graph of (15) we see that there is but one real root.[1]

If the elongation $z - z_0$ of maximum twist be measured ξ comes readily by successive approximations from

$$\frac{2\xi - 1}{\xi} = \frac{\tan \alpha_0}{\tan \alpha} + (2 - \xi) \tan \alpha \tan \alpha_0 - \tan^2 \alpha_0. \tag{16}$$

[1] Supposing $\xi = \frac{8}{10}$, its value when Poisson's ratio is $\frac{1}{4}$, the real root lies between $\frac{6}{5} \tan \alpha_0$ and $\frac{4}{3} \tan \alpha_0$.

Also from (5), if $\varphi = \varphi_0$,

$$\frac{A}{B} = \xi = \frac{\cos\alpha}{\sin\alpha} \cdot \frac{\cos\alpha_0 - \cos\alpha}{\sin\alpha - \sin\alpha_0} = \frac{\tan\frac{1}{2}(\alpha + \alpha_0)}{\tan\alpha}. \qquad (17)$$

From A, B and $\frac{A}{B}$ we get

E = Young's modulus,

n = rigidity, or slide modulus,

and

η = Poisson's ratio = ratio of lateral contraction to longitudinal extension.

Thus, if a is the radius of the cross section of the wire,[1]

$$A = \tfrac{1}{4}\pi a^4 n, \quad \text{whence} \quad n = \frac{2A}{\pi a^4}; \qquad (18)$$

$$B = \tfrac{1}{4}\pi a^4 E, \quad \text{whence} \quad E = \frac{4B}{\pi a^4}; \qquad (19)$$

$$2n(1+\eta) = E, \quad \text{whence} \quad \eta = \frac{B}{A} - 1; \qquad (20)$$

and

$$0 < \eta < \tfrac{1}{2}, \quad \text{whence} \quad \tfrac{2}{3} < \xi < 1. \qquad (21)$$

3. *Methods of Observing and Apparatus.*—Most of the commercial helical springs are "closed"; that is, the coils of wire are in close contact and sometimes require considerable force to separate them. Before "opening," that is, before giving the spring a permanent set which leaves the coils open, or apart, when hanging under no load, the radius of the cross section of the wire, a, can be found by measuring, with an ordinary standard scale, the lengths of several different sets (in hundreds) of coils. The radius of the cylinder, r_0, which envelopes the elastic central line of the wire, when hanging under no load, can be found if we know the external diam-

[1] The modulus of twist is equal to the modulus of rigidity times the polar moment of inertia of the circular cross section of the wire; and the modulus of bending is equal to Young's modulus (or the engineer's modulus) times the moment of inertia of the same cross section with respect to its diameter. For the relation between E, n and η see, for example, section 3 of Saint-Venant's edition of Clebsch, Théorie de l'Élasticité des Corps Solides, Paris, 1883.

eter of the helix and the radius of the wire. The external diameter of the helix was measured with a Brown and Sharpe micrometer screw-gauge. Thus

$$r_0 = \tfrac{1}{2} \text{ (external diameter of helix} - \text{diameter of wire).}$$

In order to avoid the uncertainty as to where the helix began and ended small dots of india ink were placed on the wire nearly in the same vertical plane, one several turns from the top and one several turns from the bottom. The number of turns from one dot to the other multiplied by 2π gives φ_0. The ends of all the springs were annealed and drawn out straight, thus bringing these ends into the axis of the cylinder of the helix. The method of suspension was to clamp the straight part of the wire in a vise which was fastened to a stable wall. By means of the micrometers of the telescopes of a Geneva cathetometer of precision used in the work it was possible to determine the exact positions of the dots with respect to one another, and thus obtain a correction to the value of φ_0 mentioned above due to the fact that the dots were not usually quite in the same vertical plane.

The vertical distance between the dots was determined by setting the movable parallel wires of the cathetometer telescopes on the upper and lower dots respectively and then turning the shaft, upon which the telescopes are mounted, until a standard meter of white bronze, which was mounted beside the spring, came into view. This meter is divided into millimeters and was made vertical by means of a plumb line. Although we do not know the absolute length of this meter an examination has shown that there are no graduation errors large enough to affect the work here considered.

A hollow brass cylinder with a three-jaw chuck enabled me to apply any load P up to six kilograms. Shot of small size were used to secure loads of the desired amounts.

The method of finding the position of maximum twist was as follows: On the straightened part of the lower end of the helix a small cork was slipped. Through the cork a straight piece of fine wire was thrust, thus forming an index. The cork was twisted until the index pointed to a plumb line suspended nearby. The empty cylinder was then clamped on the wire protruding through

the cork and in every case the index indicated a twist in accord with the law defined by equation (14). The index was then moved back until it again pointed to the plumb-line. The cylinder was then gradually loaded with shot, the index meanwhile being turned back successively to the plumb line. The rate of twist is quite rapid at first but as the position of maximum twist is approached the rate becomes less and less. The amount of twist was small in every instance, in no case did it exceed 40°. However, a considerable error in the elongation of maximum twist produces only a slight error in the value of ξ as shown by equation (16).

To find the position of no twist the index is not moved from its original position, in which it points to the plumb line, but shot are added until the index again points to the plumb line.

Thus, knowing φ_0, r_0, z_0, and z, we find a_0, s, and a from the following formulas:

$$\tan a_0 = z_0/\varphi_0 r_0, \quad s = z_0/\sin a_0,$$
$$\sin a = z/s, \quad s = \varphi_0 r_0/\cos a_0. \tag{22}$$

4. *Specimen of Computations of Constants of a Spring.*—For a copper-plated steel spring the following data were observed: diameter of wire 0.1223 cm., external diameter of helix 0.8515 cm., number of turns 295, mean axial length of helix under no load 39.615 cm., axial length at position of maximum twist 54.250 cm., axial length at position of zero twist 68.955 cm., produced by load $P = 5495.9$ grams.

(*a*) Computation of r_0, a_0 and s, with check for s.

$r_0 = \frac{1}{2}(0.8515 - 0.1223) = 0.3646$ cm.

Log 295	= 2.46982	
Log 2π	= 0.79818	
Log φ_0	= 3.26800	$\varphi_0 = 1853.5$
Log r_0	= 9.56182	
Log $\varphi_0 r_0$	= 2.82982	
Log z_0	= 1.59786	
Log tan a_0	= 8.76804	$a_0 = 3°21'17.3''$
Log sin a_0	= 8.76730	
Log z_0	= 1.59786	

$$\begin{aligned}
\text{Log } s &= 2.83056 \quad s = 676.96 \\
\text{Log cos } a_0 &= 9.99926 \\
\text{Log } \varphi_0 r_0 &= 2.82982 \\
\text{Log } s &= 2.83056 \quad s = 676.96
\end{aligned}$$

(b) Computation of a, $\xi = A/B$ and η from position of zero twist and maximum twist.

$$\begin{aligned}
\text{Log } 68.955 &= 1.83857 \\
\text{Log } s &= 2.83056 \\
\text{Log sin } a &= 9.00801
\end{aligned}$$

$$\begin{aligned}
a &= 5^\circ 50' 47.0'' \\
a_0 &= 3\ 21\ 17.3 \\
a + a_0 &= 9\ 12\ 04.3 \\
a - a_0 &= 2\ 29\ 29.7 \\
\tfrac{1}{2}(a + a_0) &= 4\ 36\ 02.2 \\
\tfrac{1}{2}(a - a_0) &= 1\ 14\ 44.8
\end{aligned}$$

$$\begin{aligned}
\text{Log tan } \tfrac{1}{2}(a + a_0) &= 8.90563 \\
\text{Log tan } a &= 9.01028 \\
\text{Log } \xi = A/B &= 9.89535 \quad A/B = 0.7859
\end{aligned}$$

$$\begin{aligned}
\text{Log } 54.250 &= 1.73440 \\
\text{Log } s &= 2.83056 \\
\text{Log sin } a &= 8.90384 \quad a = 4^\circ 35' 47.9'' \\
\text{Log tan } a &= 8.90524 \\
\text{Log tan}^2 a &= 7.81048 \quad \tan^2 a = 0.00646
\end{aligned}$$

$$\begin{aligned}
\text{Log tan } a_0 &= 8.76804 \\
\text{Log tan } a &= 8.90524 \\
\text{Log tan } a_0/\tan a &= 9.86280 \quad \tan a_0/\tan a = 0.72912.
\end{aligned}$$

$$\begin{aligned}
\text{Log tan } a \tan a_0 &= 7.67328 \\
\text{Log } (2 - \xi) &= 0.08408 \\
\text{Log } (2 - \xi) \tan a \tan a_0 &= 7.75736 \quad (2 - \xi) \tan a \tan a_0 = 0.00572.
\end{aligned}$$

$$0.72912 - 0.00646 + 0.00572 = 0.72838.$$

Therefore

$$2\xi - 1 = 0.72838\,\xi,$$

or

$$1.2716\,\xi = 1.$$

Now

$$1/\xi - 1 = B/A - 1 = \eta = 0.2716$$

and

$$\xi = A/B = 0.7864.$$

(*c*) Computation of A.

Log 2	= 0.30103	
Log sin $\frac{1}{2}(\alpha - \alpha_0)$	= 8.33728	Log (sin α − sin α_0).
Log cos $\frac{1}{2}(\alpha + \alpha_0)$	= 9.99860	
Log sec$^2\alpha$	= 0.00453	
Log φ_0^2/s^2	= 0.87488	
colog P	= 6.25996	
Log $1/A$	= 5.77628	$1/A$ = 0.000059742.
Log A	= 4.22372	A = 16739.

(*d*) Computation of B, E and n.

Log $1/A$ = 5.77634 (Mean of five measures.)
Log A/B = 9.89570
Log $1/B$ = 5.67204

Log B	= 4.32796	B = 21280.
Log 4	= 0.60206	
Log $4B$	= 4.93002	
Log πa^4	= 5.64274	
Log E	= 9.28728	E = 1937700000.
Log E	= 9.28728	$n = \frac{1}{2}\xi E$.
Log ξ	= 9.89570	
Log ξE	= 9.18298	
Log 2	= 0.30103	
Log n	= 8.88195	n = 762000000.

(*e*) Computation of $\frac{\Delta\varphi}{\varphi_0}$ at position of maximum twist.

$$x = (1-\xi)/(1+\xi) \qquad \xi = 0.7865.$$

$$\begin{aligned} \text{Log } (1-\xi) &= 9.32940 \\ \text{Log } (1+\xi) &= \underline{0.25300} \\ \text{Log } x &= 9.07740 \end{aligned}$$

$$\tfrac{1}{2}(3\alpha + \alpha_0) = 8°34'20''.5$$

$$\tfrac{1}{2}(\alpha - \alpha_0) = 0°37'15''.3$$

$$2\alpha = 9°11'35''.8$$

$$\begin{aligned} \text{Log } x &= 9.07740 \\ \text{Log sin } \tfrac{1}{2}(3\alpha+\alpha_0) &= \underline{9.17336} \\ \text{Log } x \text{ sin } \tfrac{1}{2}(3\alpha+\alpha_0) &= 8.25076 \qquad x \text{ sin } \tfrac{1}{2}(3\alpha+\alpha_0) = 0.017814 \\ \text{Log sin } \tfrac{1}{2}(\alpha-\alpha_0) &= 8.03490 \qquad \text{sin } \tfrac{1}{2}(\alpha-\alpha_0) = 0.010836 \end{aligned}$$

$$x \text{ sin } \tfrac{1}{2}(3\alpha+\alpha_0) - \text{sin } \tfrac{1}{2}(\alpha-\alpha_0) = 0.006978$$

$$\begin{aligned} \text{Log sin } \tfrac{1}{2}(\alpha-\alpha_0) &= 8.03490 \\ \text{Log } 2 &= 0.30103 \\ \text{Log } 0.006978 &= \underline{7.84373} \\ \text{Log numerator} &= 6.17966 \end{aligned}$$

$$\begin{aligned} \text{Log } x &= 9.07740 \\ \text{Log cos } 2\alpha &= \underline{9.99438} \\ \text{Log } x \text{ cos } 2\alpha &= 9.07178 \end{aligned}$$

$$1 + x \text{ cos } 2\alpha = 1.1180$$

$$\text{Log } (1 + x \text{ cos } 2\alpha) = \text{Log denominator} = 0.04844$$

$$\text{Log } \frac{\Delta\varphi}{\varphi_0} = 6.13122 \qquad \frac{\Delta\varphi}{\varphi_0} = 0.00013528.$$

(f) Table of values for $\Delta\varphi/\varphi_0$ and $\Delta\varphi$ for different values of a.

a	$\Delta\phi/\phi_0$	$\Delta\phi$
$a_0 = 3°21'17.3''$	+ 0.00000000	+ 0.0°
3 59 17.3	+ 0.00008635	+ 9.2
4 17 17.3	+ 0.00012701	+ 13.5
4 35 47.9	+ 0.00013528	+ 14.4
5 00 17.3	+ 0.00012047	+ 12.8
5 25 17.3	+ 0.00005711	+ 6.1
5 50 47.3	— 0.00002802	— 3.0
7 00 17.3	— 0.00037801	— 40.2

5. *Tables of Observed and Derived Results.*—The following tables give the numerical results appertaining to seven steel and three brass springs subjected to examination.

Table I. gives the observed data for these springs. The quantities measured are all linear. They are expressed by five significant figures; and it is believed that the precision of the measurements is correctly indicated by that number of figures.

Table II. gives the computed constants φ_0, a_0 and s for the springs. As explained above (section 3), the value of φ_0 is 2π times the number of turns of the helix used, plus a small correction for the excess or defect of that number of turns from a round number. The precision of the data here given is also supposed to be represented without exaggeration by the figures used.

Table III. gives the observed and mean values of the modulus of twist A, and the ratio of the latter to the modulus of bending B or A/B. The probable errors attached to the mean values are derived from the discrepancies between the individual values and their mean for any spring, equal weights being assigned to the individual values. The values of A/B were derived mostly from the observed elongations of maximum twist ($\Delta\varphi$ = a max.). A few values were derived from the observed elongations of zero twist ($\Delta\varphi = 0$). Such values are designated by the letter b in parenthesis following them. Apparently, the results by these two methods are of about equal weight, and hence no distinction is made among them.

Table IV. gives the computed values of the modulus of bending B, the modulus of rigidity n, Young's modulus E, and Poisson's

ratio η. The values of E and n are given in dynes per square centimeter, the value of the normal acceleration at the place of observation, latitude 40°48′28″, being $g = 980.2^{cm}/(\text{sec.})^2$.

TABLE I.

Observed Data of Springs.

Number and Metal of Springs.	Radius of Cross-Section of Wire. a	Radius of Unstrained Helix. r_0	Unstrained Length of Helix. z_0	Strained Length of Helix. z	Attached Mass. m
	Centimeter.	Centimeter.	Centimeter.	Centimeter.	Grams.
1. Copper-plated steel.	0.05680	0.32305	41.328	72.586	5408.8
				72.502	5390.2
				72.529	5392.9
				72.727	5427.4
				72.606	5403.3
				72.500	5385.9
2. Steel.	0.05605	0.32580	39.032	69.407	5756.2
				69.524	5773.6
				69.675	5802.0
				69.784	5818.5
				69.780	5816.2
3. Copper-plated steel.	0.06092	0.36430	38.153	44.273	1248.0
				46.731	1748.0
				49.174	2248.0
				51.612	2748.0
				53.593	3154.0
				56.030	3654.0
				56.049	3654.0
				60 949	4654.0
				65.850	5654.0
				66.605	5804.0
4. Copper-plated steel.	0.06115	0.36460	39.615	68.642	5444.8
				68.955	5495.9
				69.009	5504.2
				68.672	5438.9
				68.704	5440.9
5. Steel.	0.09325	1.09975	33.327	58.650	2378.1
				59.515	2459.0
				59.575	2466.0
				59.352	2453.5
				59.483	2456.8
				59.143	244[illegible].0

TABLE I.—*Continued.*

Observed Data of Springs.

Number and Metal of Springs.	Radius of Cross-Section of Wire. a	Radius of Unstrained Helix. r_0	Unstrained Length of Helix. z_0	Strained Length of Helix. z	Attached Mass. m
	Centimeter.	Centimeter.	Centimeter.	Centimeter.	Grams.
6. Steel.	0.10328	1.10480	23.689	43.529	3384.3
				43.820	3402.2
				43.496	3348.3
				43.406	3332.0
				43.407	3333.3
7. Steel.	0.15315	1.8395	30.621	35.895	2070.0
				36.478	2298.0
				41.855	4368.1
				42.343	4452.0
8. Brass.	0.06526	0.4174	57.135	63.650	341.2
				80.890	1246.2
				82.753	1338.5
				88.940	1660.9
9. Brass.	0.08239	0.5716	58.050	63.000	341.2
				76.211	1246.2
				89.380	2135.5
				92.587	2347.8
10. Brass.	0.10280	0.8494	52.860	57.444	341.2
				70.030	1246.2
				71.106	1338.5
				84.245	2163.9

TABLE II.

Computed Constants of Springs.

Number of Spring.	Angle Between Ends of Spring. ϕ_0	Slope of Unstrained Spring. a_0	Length of Spring Along Elastic Central Line s.
	Radians.	° ′ ″	Centimeter.
1	2054.6	3 33 46.5	665.03
2	1847.3	3 42 38.2	603.12
3	1690.2	3 32 43.0	616.99
4	1853.5	3 21 17.3	676.96
5	955.04	1 58 43.2	964.92
6	596.90	2 03 26.5	659.90
7	307.88	3 05 41.5	1191.10
8	2600.9	3 00 45.0	1087.20
9	1947.8	2 59 05.0	1114.90
10	1400.9	2 32 36.7	1191.10

TABLE III.

Observed and Mean Values of A and A|B.

Number of Spring.	Individual Values of *A*.	Mean Value of *A*.	Individual Values of *A|B*.	Mean Value of *A|B*.
1	11910		.7781	
	11900		.7776	
	11900		.7751	
	11890		.7761	
	11900	11900 ± 2.1	.7770	.7768 ± .00036
2	12020		.7794	
	12010		.7789	
	12010		.7783	
	12000	12008 ± 2.5	.7778	
			.7779	
			.7786 (*b*)	.7785 ± .00017
3	16660		.7798	
	16640		.7792	
	16650		.7789	
	16650		.7799	
	16660		.7794	
	16650		.7793	
	16640		.7814	
	16620		.7781	
	16590		.7776	
	16570	16633 ± 6.5	.7806	.7794 ± .00024
4	16760		.7872	
	16740		.7859	
	16730		.7856	
	16730		.7871	
	16720		.7869	
		16736 ± 4.5	.7864 (*b*)	.7865 ± .00018
5	92120		.7835	
	92090		.7794	
	92150		.7791	
	92473		.7802	
	92130		.7795	
	91830	92130 ± 51	.7799 (*b*)	.7803 ± .00045
6	137000		.7715	
	135700		.7716	
	136400		.7697	
	135700		.7724	
	135700		.7723	
		136120 ± 260	.7723 (*b*)	.7716 ± .00028

TABLE III.—*Continued.*

Observed and Mean Values of A and A/B.

Number of Spring.	Individual Values of A.	Mean Value of A.	Individual Values of A/B.	Mean Value of A/B.
7	752000		.7845	
	751500		.7832	
	743900		.7814	
	743000	747600 ± 1600	.7754	.7811 ± .0013
8	9914		.7489	
	9873		.7437	
	9855		.7510	
	9854		.7432	
			.7496	
			.7471	
			.7427	
		9874 ± 9	.7442	.7463 ± .0007
9	25100		.7448	
	24970		.7462	
	24760		.7509	
	24670		.7486	
			.7420	
		24875 ± 66	.7433	.7460 ± .0009
10	65410		.7500	
	62270		.7434	
	62860		.7414	
	61430		.7468	
			.7471	
			.7449	
			.7498	
			.7419	
			.7424	
		61742 ± 350	.7437	.7452 ± .0006

TABLE IV.

Computed Values of B, E, n and η.

Number and Metal of Spring.	Modulus of Bending. B	Young's Modulus. E	Modulus of Rigidity. n	Poisson's Ratio. η
		Dynes per Square Centimeter.	Dynes per Square Centimeter.	
1. Steel.	15320± 7.6	(1837±0.9)×10^9	(7134± 1.3)×10^8	0.2873±6
2. "	15420± 4.6	(1951±0.6)×10^9	(7592± 2.0)×10^8	.2845±3
3. "	21340±10.6	(1933±0.9)×10^9	(7533± 2.9)×10^8	.2830±4
4. "	21280± 7.5	(1899±0.7)×10^9	(7469± 2.0)×10^8	.2715±3
5. "	118100±94	(1949±1.6)×10^9	(7604± 4.2)×10^8	.2816±7
6. "	176400±343	(1939±3.8)×10^9	(7481±14.3)×10^8	.2960±5
7. "	957100±2594	(2171±5.8)×10^9	(8479±18.1)×10^8	.2802±22
8. Brass.	13230±17	(911±1.2)×10^9	(340± 0.3)×10^9	.3400±13
9. "	33340±97	(904±2.6)×10^9	(337± 0.9)×10^9	.3405±16
10. "	82850±474	(938±5.4)×10^9	(345± 2.0)×10^9	.3419±10

6. *Precision of the Constants Derived.*—From equation (8), in view of the small value of $\Delta\varphi/\varphi_0$, it is seen that so far as errors of observation are concerned the modulus of twist may be defined by

$$A = \frac{Ps^3 \cos^2 a}{(z - z_0)\varphi_0^2}.$$

Since $\cos^2 a = 1 - (z/s)^2$, and since z/s is small for each of the springs, it appears that the relative errors of A from different determinations for the same spring depend essentially on the errors of P and $(z - z_0)$ only. It may be safely assumed, however, that the errors of P, and likewise those of φ_0 and s, are negligible in comparison with those of $(z - z_0)$. Thus we conclude that

$$\frac{\Delta A}{A} = -\frac{\Delta(z - z_0)}{z - z_0},$$

or that the probable error of A is the same fraction of A as the probable error of $(z - z_0)$ is of it. We believe, therefore, that the probable errors of the mean values of A are essentially correct as given in Table III.

From similar reasoning we conclude that the probable errors of the ratios A/B given in Table III. afford trustworthy indications of the precision of the mean values of those quantities.

For computing the probable errors of B, E, n, and η from those of the independently observed quantities A and A/B, it is essential to note the relations expressed by equations (18) to (20) in connection with the two following equations. Thus let

$$A = u, \quad \text{and} \quad \frac{A}{B} = \xi,$$

wherein u and ξ are supposed to be observed quantities. Then

$$B = \frac{u}{\xi},$$

and if δB, δu, and $\delta\xi$ denote the probable errors of B, u and ξ, respectively,

$$\left(\frac{\delta B}{B}\right)^2 = \left(\frac{\delta u}{u}\right)^2 + \left(\frac{\delta\xi}{\xi}\right)^2.$$

Hence from the values of δu and $\delta\xi$ given in Table III. the probable errors of B in Table IV. are derived.

From equation (18) it is seen that the probable error of n is the same fraction of n as the probable error of A is of it; and similarly from (19) the probable error of E is the same fraction of E as the probable error of B is of it.

Finally, from equation (20) it appears that the probable error of η is equal to the probable error of the reciprocal of A/B; and this is the same as $(B/A)^2$ times the probable error of A/B.

It will be observed from Table IV. that the precision is highest, and of about the same order, for the first four steel springs; the probable errors of A and n being about 1/3500 part, those of B and E about 1/2000 part, and those of η about 1/700 part. This higher precision is most likely due to the fact that the springs in question were the best as to uniformity, temper and finish of all the springs used. Next in order of precision are the constants for the steel springs Nos. 5 and 6. These are inferior in quality to Nos. 1–4 as regards uniformity and finish. Steel spring No. 7 is markedly different from the others in being "open-wound," of large diameter, of heavier wire than the others, and relatively short in axial length z_0. The means at hand did not permit a sufficient elongation of this spring to attain results of higher precision for A, and the ratio A/B was also less definitely measurable than in the cases of the other springs.

The precision is relatively low, as would be expected, for the constants of the brass springs, although they are surprisingly uniform as to slope and diameter for commercial products. The well-known instability of brass as compared with steel is undoubtedly the source of this lower order of precision.

7. *Summary of Conclusions.*—From the details of the investigation given above the following conclusions appear to be established:

(*a*) The static theory of helical springs as expressed by equations (1) to (4), and given for the first time by Kelvin and Tait in the first edition of their Natural Philosophy, 1867, has been experimentally verified. Starting from this theory the above paper shows that:

(*b*) To a high order of approximation the force required to elongate a helical spring varies directly as the product of the elongation of the spring and the square of the secant of its slope. The exact law for finite elongations of any such spring is expressed in a simple form by equations (7), (8) and (14).

(*c*) When such springs are elongated by an axial pull, with no applied torsional couple, they undergo a twist about the axis of the helix. This twist increases as the pull increases up to a maximum, and thereafter decreases continuously with increasing pull.

(*d*) The law connecting the elongation of a spring with the applied force furnishes a ready method of finding the modulus of twist and hence the modulus of rigidity of the wire of the spring.

(*e*) Observations of the elongations of maximum twist and zero twist furnish independent methods of deriving the ratio of the modulus of twist to the modulus of bending.

(*f*) By a combination of the results furnished by (*d*) and (*e*) the modulus of bending, Young's modulus, and Poisson's ratio are readily derived.

(*g*) By the methods explained it is easily practicable to attain, in the case of a well-made steel spring, a precision indicated in round numbers by a probable error of one part in three thousand for the modulus of twist, of one part in two thousand for Young's modulus and of one part in five hundred for Poisson's ratio.

8. *Acknowledgments.*—The author desires to express his obligations to the officers of the Department of Physics as well as to those of the Department of Mechanics of Columbia University for aid at all times cordially rendered during the progress of this investigation. He is especially indebted to Professor R. S. Woodward for assistance in developing the mathematical theory of the springs and for suggestions with reference to the reduction and discussion of the observations.

COLUMBIA UNIVERSITY,
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ON THE BOUNDARY CONDITIONS OF THE ELECTRICAL FIELD.

By William Duane.

1. In the method of presenting the modern theory of electricity suggested first by Heaviside,[1] and several years later by Hertz,[2] the laws and definitions represented by the following equations are taken as a basis, and from them are deduced many of the laws and facts that are suitable to direct experimental verification. The fundamental equations are :

$$-\operatorname{curl} P_e = 4\pi i_m \tag{1}$$

$$\operatorname{curl} P_m = 4\pi i_e \tag{2}$$

$$E_e = \frac{K}{8\pi} P_e^2 \tag{3}$$

$$E_m = \frac{\mu}{8\pi} P_m^2 \tag{4}$$

$$i_m = \frac{1}{4\pi} \frac{d\mu P_m}{dt} \tag{5}$$

$$i_e = \frac{1}{4\pi} \frac{dKP_e}{dt} + CP_e \tag{6}$$

in which the letters represent the quantities printed opposite to them in the following table.

P_e = electric force.
P_m = magnetic force.
i_e = electric current density.
i_m = magnetic current density.
E_e = electric energy density.
E_m = magnetic energy density.

[1] Electrician, 1885.
[2] Wied. Ann., 36, p. 1, 1889.

K = electric inductivity.
μ = magnetic inductivity.
C = electric conductance.

For points where there is impressed electric or magnetic force equations 3, 4, 5 and 6 must be modified slightly. The modifications, however, do not affect the following conclusions.

2. In reasoning from these equations it is customary to deduce the usual expressions for the electric, magnetic and vector potentials, and from them the general laws and the phenomena. The author has found some difficulty in following the reasoning without making special assumptions as to the character of the electromagnetic field at the boundary of the electrical system, or at infinity if the system is not bounded. In fact it would appear impossible to prove the relations between the potentials and the densities of electricity, magnetism and current without some boundary condition or its equivalent.

3. In the electrostatic case the line of reasoning is somewhat as follows: Since the currents are zero, curl $P_e = 0$, and hence the line integral of the electric force taken from any point P to a fixed point O is independent of the path along which the integration is performed, *i. e.*, is a function of the coördinates of P only. The value of the potential V at P is defined to be this line integral, the potential at O being chosen as the zero of potential. From this it follows that the electric force at any point is minus the space derivative of the potential at the point taken in the direction of the force, and if X, Y and Z are the rectilinear components of P_e,

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}.$$

The density e of free electricity is defined to be the divergence of the electric force divided by 4π or in symbols

$$e = \frac{1}{4\pi}\,\text{div.}\, P_e = -\frac{1}{4\pi}\nabla^2 V.$$

In order to deduce an expression for V in terms of e, it is first shown that if e is given at every point in space the potential can have but one value at each point, the potential at O being zero, as follows:

Assume there are two values V_1 and V_2 of the potential at every point, and let $\varphi = V_1 - V_2$. In Green's Theorem substitute φ for both the functions, and apply the theorem to the space inside a sphere of radius ρ, the center of which is in or near the electrical field. Thus :

$$\int \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] d\tau$$

$$= \int \varphi \frac{\partial \varphi}{\partial n} dS - \int \varphi \nabla^2 \varphi \, d\tau. \qquad (7)$$

The last volume integral vanishes because

$$-4\pi e = \nabla^2 V_1 = \nabla^2 V_2$$

and therefore

$$\nabla^2 \varphi = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

everywhere. If the surface integral approaches zero as the spherical bounding surface is taken larger and larger, the first volume integral taken throughout all space is zero. Hence since it is a sum of terms each of which is positive, each term must be zero itself, and

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial z} = 0.$$

φ is therefore independent of x, y and z, and since at the point of zero potential $\varphi = 0$, φ must be zero everywhere, and $V_1 = V_2$. Hence if the above surface integral vanishes for $\rho = \infty$, the potential can have but one value at each point in space, the density of free electricity being given everywhere.

4. There is nothing, however, in the fundamental equations (1)–(6) to indicate that this surface integral vanishes. In order to prove the solution unique it is necessary that some boundary condition or the equivalent should be given. In fact the potential is not completely determined by the value of

$$e = \frac{-1}{4\pi} \nabla^2 V,$$

and the potential at any point of reference O; for any function of the form

$$f(xyz) = Ax + By + Cz + D$$

in which A, B, C and D are constants, can be added to V, and, if the constants are determined so that f is zero at the point O, $V + f$ satisfies the equation

$$e = \frac{-1}{4\pi} \nabla^2(V + f)$$

and the conditions at O.

5. The gap in the reasoning may be filled in some such way as the following: Assume that the electrical system has a finite quantity of energy; an assumption that evidently is true in all actual cases. The quantity of electrical energy in a spherical shell of thickness $d\rho$ and radius ρ is

$$dE_e = \frac{d\rho}{8\pi} \int K P_e^2 dS,$$

the integration extending over the surface of the sphere. The whole energy is

$$E_e = \frac{1}{8\pi} \int_0^\infty d\rho \int K P_e^2 dS.$$

Since the surface integral is always positive, and E_e is finite, the surface integral expressed as a function of ρ must vanish for $\rho = \infty$. The surface integral may be expressed in the form

$$\int K P_e^2 \rho^2 d\omega,$$

in which $d\omega$ is the solid angle subtended at the center of the sphere by dS. K is always positive (for energy is never negative), and becomes unity for points in free space. Hence, as the integral is the sum of positive terms, each term must vanish for $\rho = \infty$, and the electric force P_e must vanish at infinity in such a way that $\rho P_e = 0$, for $\rho = \infty$. This then is the condition equivalent to a boundary condition that the electric forces must fulfill at infinity.

6. From equations (1)–(6) we cannot tell whether the potential has the same or different values at points on the infinite sphere; but a condition for the potential may be obtained from the finiteness of the energy as follows: The difference of potential between any two points P and P' on the sphere of radius ρ is

$$V_{P'} - V_P = \int_{P'}^{P} P_e \cos \varepsilon \, dl,$$

dl being the element of length of the great circle passing through P and P', and ε, the angle between dl and P_e. If $d\varphi$ is the angle subtended at the center of the sphere by dl, $dl = \rho d\varphi$ and

$$V_{P'} - V_P = \int_{P'}^{P} P_e \cos \varepsilon \, \rho d\varphi.$$

The integral is numerically less than or equal to the integral

$$\int_{P'}^{P} \overline{P}_e \, \rho d\varphi$$

in which $\overline{P}_e$ is the maximum value of P_e on the great circle. Hence

$$V_P - V_P \leqq \int_{P'}^{P} \overline{P}_e \, \rho d\varphi$$

or

$$V_{P'} - V_P \leqq \overline{P}_e \rho\varphi,$$

φ being the angle subtended by the radii drawn from the center of the sphere to P and P' respectively. Since this angle cannot be greater than 2π, and $\overline{P}_e \rho$ approaches zero as ρ increases, we have

$$V_{P'} - V_P = 0 \quad \text{for} \quad \rho = \infty,$$

i. e., the potential at all points on the infinite sphere has the same value.

7. In a similar manner it can be proved that the potential at infinity is finite, the potential at any point O being zero. If we integrate along a straight line from O to infinity the potential at infinity is

$$V_\infty = -\int_0^\infty P_e \cos \varepsilon \, d\rho.$$

Since $\cos \varepsilon$ is never numerically greater than one, V_∞ is numerically less than or equal to

$$\int_0^\infty P_e d\rho.$$

P_ρ is never infinite, and vanishes for $\rho = \infty$ more rapidly than $\frac{A}{\rho}$, A being a constant. Hence the integral is finite and V_∞ is finite.

8. It follows from these conditions for the potential that as ρ increases the surface integral in equation (7) approaches the value

$$\varphi_\infty \int \frac{\partial \varphi}{\partial n} dS$$

in which

$$\varphi_\infty = V_{1\infty} - V_{2\infty},$$

and is finite and constant. By a well-known theorem (a particular case of Green's theorem)

$$\int \frac{\partial \varphi}{\partial n} dS = \int \nabla^2 \varphi d\tau.$$

These integrals are zero, since $\nabla^2 \varphi = 0$ everywhere. Hence as ρ increases the surface integral approaches zero, and V is completely determined by the values of e, and the conditions at infinity.

9. Heaviside's proof[1] that the electric field is uniquely determined by the values of e is expressed in terms of vector analysis. If the two vectors R and R' have no curl and the same divergence, their geometric difference $R - R'$ has zero curl and zero divergence everywhere. It does not follow from this, however, that the vector $R'' = R - R'$ is zero. Any vector, for instance, that is constant in magnitude and direction has zero curl and zero divergence and is not zero itself. Nor is the simple statement of the condition "if R'' vanishes at infinity" sufficient, for, in the first place, no proof is given that R'' vanishes at infinity, and in the second, it does not follow from the vanishing of R'' at infinity that its lines must form closed curves. The truth of the latter is evident in the simple case of the electric field of force due to a single charge.

10. That R'' is zero, however, follows immediately from the conditions that the potential has the same value at all points on the infinite sphere. As Heaviside says, "the lines of R'' can neither form closed curves nor end at any point." Further, they cannot run off in both directions to infinity, for if they did the line integral of

[1] "Electrical Papers," Vol. I., p. 217.

R'' taken along any one of them would not be zero, and the above-stated conditions for the potential would not be fulfilled. Hence R'' must be zero.

11. We know from the theory of the Newtonian potential function that the expression

$$\varphi = \int \frac{e d\tau}{r}$$

satisfies the equation

$$\nabla^2 \varphi = -4\pi e$$

and that it vanishes at infinity. Hence $\varphi + k$ approaches the constant value k at infinity, and is the only function that can represent the potential. The value of k depends upon the point chosen as the zero of potential.

12. In the general case of an electro-magnetic field in which the electric current and the free magnetism are not necessarily zero, the magnetic force P_m satisfies the equations

$$\text{curl } P_m = 4\pi i_e, \tag{2}$$

$$\text{div. } P_m = 4\pi m; \tag{8}$$

equation (8) being the definition of the density of free magnetism m. If i_e and m are given these two equations do not determine P_m completely, as any vector whose curl and divergence are zero (for instance one constant in magnitude and direction) can be added to P_m and the resulting vector satisfies equations (2) and (9). We can find, however, a third condition that must be satisfied by the magnetic force by reasoning from the finiteness of the magnetic energy. The condition is as before, that P_m must vanish at infinity in such a way that $\rho P_m = 0$ if $\rho = \infty$. This holds even in variable states. In these there may be radiation of energy in any or all directions, but it can be proved from the fundamental equations that the energy travels with finite velocity, and hence the radiation does not affect the conditions at infinity.

13. The condition $\rho P_m = 0$, if $\rho = \infty$, together with equations (2) and (8), if i_e and m are given, can be shown to determine P_m completely as follows. We know from the Newtonian potential theory that the vector P_m', the rectilinear components of which are

$$\alpha' = -\frac{\partial}{\partial x}\int\frac{m d\tau}{r}, \quad \beta' = -\frac{\partial}{\partial y}\int\frac{m d\tau}{r}, \quad \gamma' = -\frac{\partial}{\partial z}\int\frac{m d\tau}{r} \qquad (9)$$

satisfies the equations (8) and

$$\text{curl } P_m' = 0,$$

and that P_m' vanishes at infinity in such a way that $\rho P_m' = 0$, if $\rho = \infty$. From the vector potential theory we know that P_m'', the components of which are

$$\alpha'' = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad \beta'' = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad \gamma'' = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \qquad (10)$$

in which F, G and H are the integrals

$$F = \int\frac{u d\tau}{r}, \quad G = \int\frac{v d\tau}{r}, \quad H = \int\frac{w d\tau}{r} \qquad (11)$$

u, v and w being the rectilinear components of i_e, satisfies the equations (2) and

$$\text{div. } P_m'' = 0,$$

and that P_m'' vanishes at infinity in such a way that $\rho P_m'' = 0$, if $\rho = \infty$. Hence the sum of the two vectors

$$P_{m_1} = P_m' + P_m''$$

must fulfill the conditions $\rho P_{m_1} = 0$, if $\rho = \infty$, and must satisfy the equations (2) and (8). P_{m_1} can be shown to be equal to P_m; for their difference

$$P_{m_2} = P_m - P_{m_1}$$

must satisfy the equations (2) and (8) and the above-stated condition at infinity. From equation (2) it appears that P_{m_2}, if not zero everywhere, must have a potential φ. By using Green's theorem as before, it follows that φ must be constant, and therefore $P_{m_2} = 0$. Hence $P_m = P_{m_1}$, and the magnetic force is determined completely by the values of m, i_e and the condition at infinity, P_m being given by equations (9), (10), (11) and

$$P_m = P_m' + P_m''.$$

14. The same reason is applicable to the electric force in the variable states, and it, therefore, is determined completely by the values of i_m, e and the finiteness of the energy.

HALE PHYSICAL LABORATORY, UNIVERSITY
OF COLORADO.

NOTE ON KIRCHOFF'S THEORY OF MAGNETOSTRICTION.

BY S. SANO.

THE medium treated by Kirchoff[1] in his theory of magnetostriction is supposed initially to be an isotropic elastic solid, which in the magnetized condition is completely free from hysteresis and time-lag, and to be such that the components of the intensity of magnetization in the medium are not only dependent upon the magnetizing force, but are also linear functions for the strain components. The object of the present investigation is to make a slight extension of Kirchoff's theory to the case where the susceptibility k and two strain coefficients k' and k'' are functions of the strength of the magnetizing force, the other assumptions made to the elastic medium being retained. I shall first find the expressions for the components of the intensity of magnetization in the medium, and then discuss the mechanical forces acting on the elastic solid.

Let H be the strength of the magnetizing force, and A, B, C the components of the intensity of magnetization, and u, v, w those of the displacement at a point (x, y, z) in the elastic solid, referred to rectangular coördinate axes. At first take the axis of z parallel to the direction of the magnetic force at the point (x, y, z); then A, B, C take the following forms:

$$\left.\begin{aligned} A = {} & a_1\frac{\partial u}{\partial x} + a_2\frac{\partial v}{\partial y} + a_3\frac{\partial w}{\partial z} + a_4\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ & + a_5\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + a_6\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ B = {} & b_1\frac{\partial u}{\partial x} + b_2\frac{\partial v}{\partial y} + b_3\frac{\partial w}{\partial z} + b_4\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ & + b_5\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + b_6\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \end{aligned}\right\} \quad (1)$$

[1] Sitzber. d. k. Acad. d. Wiss. zu Berlin, p. 137 (1884); Wied. Ann., Vol. XXIV. p. 52 (1885); Gesammelte Abhandlungen, Nachtrag, p. 91, Leipzig (1891).

$$C = c_0 + c_1\frac{\partial u}{\partial x} + c_2\frac{\partial v}{\partial y} + c_3\frac{\partial w}{\partial z} + c_4\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + c_5\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + c_6\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

where the a's, b's and c's are dependent on H but not on the strain components. Next introduce a new set of rectangular coördinate axes x', y', z', the axis of z' coinciding with the old axis of z. Let (A', B', C') and (u', v', w') be the intensity of magnetization and the displacement referred to new axes, then we have

$$A' = a_1'\frac{\partial u'}{\partial x'} + a_2'\frac{\partial v'}{\partial y'} + a_3'\frac{\partial w'}{\partial z'} + a_4'\left(\frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}\right) + a_5'\left(\frac{\partial w'}{\partial x'} + \frac{\partial u'}{\partial z'}\right) + a_6'\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right),$$

$$B' = b_1'\frac{\partial u'}{\partial x'} + b_2'\frac{\partial v'}{\partial y'} + b_3'\frac{\partial w'}{\partial z'} + b_4'\left(\frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}\right) + b_5'\left(\frac{\partial w'}{\partial x'} + \frac{\partial u'}{\partial z'}\right) + b_6'\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right),$$

$$C' = c_0' + c_1'\frac{\partial u'}{\partial x'} + c_2'\frac{\partial v'}{\partial y'} + c_3'\frac{\partial w'}{\partial z'} + c_4'\left(\frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}\right) + c_5'\left(\frac{\partial w'}{\partial x'} + \frac{\partial u'}{\partial z'}\right) + c_6'\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right),$$

where the a''s, b''s and c''s have significations similar to those of the a's, b's and c's. Denoting the angle between the axes x and x' by θ, which we shall suppose to be positive when the angle between the axes y and y' diminishes by increasing θ, the new coefficients, a_1', a_2', a_3' ··· are related to the old coefficients a_1, a_2 ···, c_0, c_1 ··· by the following equations:

$$a_1' = a_1\cos^3\theta + (b_1 + 2a_6)\cos^2\theta\sin\theta + (a_2 + 2b_6)\cos\theta\sin^2\theta + b_2\sin^3\theta,$$

$$a_2' = a_2\cos^3\theta + (b_2 - 2a_6)\cos^2\theta\sin\theta + (a_1 - 2b_6)\cos\theta\sin^2\theta + b_1\sin^3\theta,$$

$$a_3' = a_3\cos\theta + b_3\sin\theta,$$

$$a_4' = a_4\cos^2\theta + (-a_5 + b_4)\cos\theta\sin\theta - b_5\sin^2\theta,$$

$$
\left.\begin{aligned}
a_5' &= a_5\cos^2\theta + (a_4+b_5)\cos\theta\sin\theta + b_4\sin^2\theta,\\
a_6' &= a_6\cos^3\theta + (-a_1+a_2+b_6)\cos^2\theta\sin\theta +\\
&\qquad (-a_6-b_1+b_2)\cos\theta\sin^2\theta - b_6\sin^3\theta,\\
&\cdots\cdots\cdots\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
c_0' &= c_0,\\
c_1' &= c_1\cos^2\theta + 2c_6\cos\theta\sin\theta + c_2\sin^2\theta,\\
c_2' &= c_2\cos^2\theta - 2c_6\cos\theta\sin\theta + c_1\sin^2\theta,\\
c_3' &= c_3,\\
c_4' &= c_4\cos\theta - c_5\sin\theta,\\
c_5' &= c_5\cos\theta + c_4\sin\theta,\\
c_6' &= c_6\cos^2\theta + (-c_1+c_2)\cos\theta\sin\theta - c_6\sin^2\theta.
\end{aligned}\right\} \qquad (2)
$$

The supposition that the medium is originally isotropic leads to the following relations among the coefficients:

$$
\begin{aligned}
a_1' &= a_1,\\
a_2' &= a_2,\\
&\cdots\cdots\\
c_6' &= c_6,
\end{aligned}
$$

whence by (2),

$$
\begin{aligned}
&a_1 = a_2 = a_3 = a_6 = b_1 = b_2 = b_3 = b_6 = c_4 = c_5 = c_6 = 0,\\
&a_5 = b_4,\\
&b_5 = -a_4,\\
&c_2 = c_1,
\end{aligned}
$$

Thus (1) may be written

$$
\left.\begin{aligned}
A &= a_4\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) + b_4\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\\
B &= b_4\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) - a_4\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\\
C &= c_0 + c_1\frac{\partial u}{\partial x} + c_1\frac{\partial v}{\partial y} + c_3\frac{\partial w}{\partial z}.
\end{aligned}\right\} \qquad (3)
$$

Putting $\quad c_0 = kH, \quad c_1 = -k'H, \quad 2b_4 = -k''H,$
$\qquad c_3 - c_1 - 2b_4 = -k'''H, \quad 2a_4 = -k''''H,$

and changing the coördinate axes, we have from (3)

$$\left.\begin{aligned} A = lkH &- lk'H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \\ &- k''H\left\{l\frac{\partial u}{\partial x}+\frac{n}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\frac{m}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \\ &- lk'''H\left\{l^2\frac{\partial u}{\partial x}+m^2\frac{\partial v}{\partial y}+n^2\frac{\partial w}{\partial z}+mn\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right. \\ &\qquad \left.+nl\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+lm\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \\ &- k''''H\left\{mn\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right)-\frac{m^2-n^2}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right. \\ &\qquad \left.-\frac{lm}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\frac{nl}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}, \end{aligned}\right\} \quad (4)$$

with analogous expressions for B and C, where l, m, n are the direction cosines of the magnetic force at the point (x, y, z).

Let α, β, γ be the components of the magnetic force, so that $\alpha = lH, \beta = mH, \gamma = nH$. The absence of hysteresis in the medium gives the following equations:

$$\frac{\partial B}{\partial \gamma}-\frac{\partial C}{\partial \beta}=\frac{\partial C}{\partial \alpha}-\frac{\partial A}{\partial \gamma}=\frac{\partial A}{\partial \beta}-\frac{\partial B}{\partial \alpha}=0. \quad (5)$$

The necessary and sufficient conditions that (4) may satisfy (5) are easily found to be

$$k'' = \frac{H}{2}\frac{dk''}{dH},$$
$$k''' = 0.$$

Hence we have finally

$$\left.\begin{aligned} A = k\alpha &- k'\alpha\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \\ &- k''\left\{\alpha\frac{\partial u}{\partial x}+\frac{\gamma}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\frac{\beta}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \\ &- \frac{\alpha}{2H}\frac{dk''}{dH}\left\{\alpha^2\frac{\partial u}{\partial x}+\beta^2\frac{\partial v}{\partial y}+\gamma^2\frac{\partial w}{\partial z}+\beta\gamma\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right. \\ &\qquad \left.+\gamma\alpha\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\alpha\beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}, \\ B = &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C = &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (6)$$

When k'' is independent of H (6) takes the forms identical with Kirchoff's corresponding equations, though k and k' may depend on H. The essential difference between the above expressions for the components of the intensity of magnetization and those of Kirchhoff is due to the presence of the terms involving the differential coefficient of k'' with respect to H. These terms represent a magnetization in the direction of the magnetic force whose intensity is

$$-\tfrac{1}{2}\frac{dk''}{dH}\left\{\alpha^2\frac{\partial u}{\partial x}+\beta^2\frac{\partial v}{\partial y}+\gamma^2\frac{\partial w}{\partial z}+\beta\gamma\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$
$$\left.+\gamma\alpha\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\alpha\beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\},$$

being proportional to the linear dilatation

$$\frac{1}{H^2}\left\{\alpha^2\frac{\partial u}{\partial x}+\beta^2\frac{\partial v}{\partial y}+\gamma^2\frac{\partial w}{\partial z}+\beta\gamma\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$
$$\left.+\gamma\alpha\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\alpha\beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}$$

of the solid in the direction of the magnetic force. Thus (6) may be considered to represent a magnetization which is obtained by superposing the magnetization given by Kirchhoff's equations upon that whose direction is parallel to that of the magnetic force and whose intensity is proportional to dk''/dH and the linear dilatation in the direction of the magnetic force, with the due coefficient of proportionality. As has been already stated, the extension of Kirchhoff's theory to cases in which k' depends on H but k'' does not, requires no change of the forms in the components of the intensity of magnetization. But this is not the case for the expressions of forces acting on the solid, as will be seen later.

The expressions for the mechanical forces acting on the elastic solid can be obtained by proceeding in a similar way as Helmholtz[1] and Kirchhoff,[2] or we may proceed in the following manner.

Let us consider a system at rest, consisting of a permanent magnet and the elastic magnetic solid separated by air. The whole

[1] Helmholtz, Wied. Ann., Vol. XIII., p. 385; Monatsber. d. Berl. Akad, v. 17, Feb., 1881; Wissenschaftliche Abhandlungen, Vol. I., p. 798, Leipzig (1882).

[2] Kirchhoff, l. c.

system is supposed to be at an uniform temperature. It is convenient to assume the continuous transition in the boundary of the different media, as is usually done in similar discussions. Considering the thin stratum of transition from the solid to the air to be a portion of the solid, we have $k = k' = k'' = 0$ at its boundary. Let a, b, c be the components of the magnetic induction at the point (x, y, z), and let $d\tau_m$, $d\tau_a$, $d\tau$ be the volume elements of the permanent magnet, the air, and the solid respectively.

Now the work done by external forces to produce the magnetic field reckoned per unit volume is

$$\frac{1}{4\pi}\int (ada + \beta db + \gamma dc),$$

where the integral is to be taken from the state $H = 0$. Hence the total work done by external forces to produce the magnetic field in the system under consideration is

$$W = \frac{1}{4\pi}\int R_m d\tau_m + \frac{1}{4\pi}\int R_a d\tau_a + \frac{1}{4\pi}\int R d\tau,$$

where R_m, R_a, R denote the values of $\int (ada + \beta db + \gamma dc)$ in the permanent magnet, the air, and the solid, respectively.

Let the element of the elastic solid at (x, y, z) make an infinitesimal virtual displacement $(\delta u, \delta v, \delta w)$, while the permanent magnet is completely fixed. The virtual displacement is supposed to take place so slowly that the temperature of the system remains unaffected. The variation of W which the system acquires during the virtual displacement is

$$\begin{aligned}\delta W &= \frac{1}{4\pi}\int \delta R_m d\tau_m + \frac{1}{4\pi}\int \delta R_a d\tau_a + \frac{1}{4\pi}\int \delta R d\tau. \\ &= \frac{1}{4\pi}\int (a\delta a + \beta\delta b + \gamma\delta c) d\tau_m \\ &\quad + \frac{1}{4\pi}\int (a\delta a + \beta\delta b + \gamma\delta c) d\tau_a + \frac{1}{4\pi}\int \delta R d\tau, \qquad (7)\end{aligned}$$

since

$$\delta R_m = a\delta a + \beta\delta b + \gamma\delta c.$$
$$\delta R_a = a\delta a + \beta\delta b + \gamma\delta c.$$

δR can not be expressed in a simple form as δR_m or δR_a.

The absence of electric current in the system warrants the existence of the magnetic potential V at the point (x, y, z). If we multiply

$$\frac{\partial \delta a}{\partial x} + \frac{\partial \delta b}{\partial y} + \frac{\partial \delta c}{\partial z} = 0$$

by V and integrate the result over the whole space we have

$$\begin{aligned}
&\int V\left(\frac{\partial \delta a}{\partial x} + \frac{\partial \delta b}{\partial y} + \frac{\partial \delta c}{\partial z}\right) d\tau_m \\
&\qquad + \int V\left(\frac{\partial \delta a}{\partial x} + \frac{\partial \delta b}{\partial y} + \frac{\partial \delta c}{\partial z}\right) d\tau_a + \int V\left(\frac{\partial \delta a}{\partial x} + \frac{\partial \delta b}{\partial y} + \frac{\partial \delta c}{\partial z}\right) d\tau \\
&= \int (a\delta a + \beta \delta b + \gamma \delta c) d\tau_m + \int (a\delta a + \beta \delta b + \gamma \delta c) d\tau_a \\
&\qquad + \int (a\delta a + \beta \delta b + \gamma \delta c) d\tau = 0.
\end{aligned} \tag{8}$$

With the aid of (8), (7) may be written

$$\delta W = \frac{1}{4\pi} \int \{\delta R - (a\delta a + \beta \delta b + \gamma \delta c)\} d\tau. \tag{9}$$

Now a, b, c and R are explicit functions of α, β, γ, k, k', k'', (and dk''/dH), and the strain components $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \cdots \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ only, while k, k', k'' are again functions of x, y, z, and H. Hence the variation δ may conveniently be considered as taking place in three different stages: During the first stage there is neither deformation nor change of position of elementary portions, so that the only varying quantities are α, β, γ, and k, k', k'', since the latter are dependent on H. During the second stage of the variation the material point at $(x - \delta u, y - \delta v, z - \delta w)$ arrives at (x, y, z) after the virtual displacement, the magnetic field and the strain components being supposed invariable. Thus it corresponds to the mere effect of the change of position without deformation. The variation of the strain components in constant field and without the change of position forms the third stage. Let these three kinds of the variations be respectively donated by δ_1, δ_2, and δ_3; then we may write

$$\delta R = a\delta_1 a + \beta \delta_1 b + \gamma \delta_1 c + \delta_2 R + \delta_3 R,$$

$$\alpha\delta a + \beta\delta b + \gamma\delta c = a\delta_1 a + \beta\delta_1 b + \gamma\delta_1 c + \delta_2(a\alpha + b\beta + c\gamma) + \delta_3(a\alpha + b\beta + c\gamma).$$

Hence (9) becomes

$$\delta W = \frac{1}{4\pi}\int [\delta_2\{R - (a\alpha + b\beta + c\gamma)\} + \delta_3\{R - (a\alpha + b\beta + c\gamma)\}d\tau \tag{10}$$

$$= -\int \delta_2 S d\tau - \int \delta_3 S d\tau,$$

where

$$S = \frac{1}{4\pi}\{(a\alpha + b\beta + c\gamma) - R\} \tag{11}$$

$$= \frac{1}{4\pi}\int (ada + bd\beta + cd\gamma).$$

No by virtue of equations (6) (11), and the well known relations

$$a = \alpha + 4\pi A,$$
$$b = \beta + 4\pi B,$$
$$c = \gamma + 4\pi C,$$

we get at once

$$S = \frac{H^2}{8\pi} + \int_0^H kHdH - \int_0^H k' HdH\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \frac{k''}{2}\left\{\alpha^2\frac{\partial u}{\partial x} + \beta^2\frac{\partial v}{\partial y} + \gamma^2\frac{\partial w}{\partial z} + \beta\gamma\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \gamma\alpha\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial z}\right) + \alpha\beta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}.^1$$

We have immediately

$$\int \delta_2 S d\tau = -\int\left\{\left(\frac{\partial S}{\partial x}\right)_H \delta u + \left(\frac{\partial S}{\partial y}\right)_H \delta v + \left(\frac{\partial S}{\partial x}\right)_H \delta w\right\} d\tau,$$

where $\left(\frac{\partial S}{\partial x}\right)_H$, $\left(\frac{\partial S}{\partial y}\right)_H$, $\left(\frac{\partial S}{\partial z}\right)_H$ represent the differential coefficients of S with respect to x, y, z, respectively, keeping α, β, γ constant during the differentiation. The suffix H implies in the following a similar signification. Also

[1] In the above-cited paper Kirchhoff denotes by G a certain expression which requires a modification, when the susceptibility k is dependent on the strength of the magnetizing force.

$$\int \delta_3 S d\tau = -\int \Big[\int_0^H k' H dH \Big(\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y} + \frac{\partial \delta w}{\partial z} \Big) + \frac{k''}{2} \Big\{ \alpha^2 \frac{\partial \delta u}{\partial x} + \beta^2 \frac{\partial \delta v}{\partial y} + \gamma^2 \frac{\partial \delta w}{\partial z} + \beta\gamma \Big(\frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y} \Big) + \gamma\alpha \Big(\frac{\partial \delta w}{\partial x} + \frac{\partial \delta v}{\partial z} \Big) + \alpha\beta \Big(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \Big) \Big\} \Big] d\tau$$

$$= \int \Big[\Big\{ \frac{\partial}{\partial x} \int_0^H k' H dH + \tfrac{1}{2} \frac{\partial}{\partial x} (k''\alpha^2) + \tfrac{1}{2} \frac{\partial}{\partial y} (k''\alpha\beta) + \tfrac{1}{2} \frac{\partial}{\partial z} (k''\gamma\alpha) \Big\} \delta u + \Big\{ \frac{\partial}{\partial y} \int_0^H k' H dH + \tfrac{1}{2} \frac{\partial}{\partial x} (k''\alpha\beta) + \tfrac{1}{2} \frac{\partial}{\partial y} (k''\beta^2) + \tfrac{1}{2} \frac{\partial}{\partial z} (k''\beta\gamma) \Big\} \delta v + \Big\{ \frac{\partial}{\partial z} \int_0^H k' H dH + \tfrac{1}{2} \frac{\partial}{\partial x} (k''\gamma\alpha) + \tfrac{1}{2} \frac{\partial}{\partial y} (k''\beta\gamma) + \tfrac{1}{2} \frac{\partial}{\partial z} (k''\gamma^2) \Big\} \delta w \Big] d\tau,$$

since $k = k' = k'' = 0$ at the surface of the solid. Hence (10) becomes

$$\delta W = -\int \Big[\Big\{ -\Big(\frac{\partial S}{\partial x} \Big)_H + \frac{\partial}{\partial x} \int_0^H k' H dH + \tfrac{1}{2} \frac{\partial}{\partial x} (k''\alpha^2) + \tfrac{1}{2} \frac{\partial}{\partial y} (k''\alpha\beta) + \tfrac{1}{2} \frac{\partial}{\partial z} (k''\gamma\alpha) \Big\} \delta u + \cdots \Big] d\tau. \tag{12}$$

If we denote the components of the mechanical force per unit of volume due to magnetism by X, Y, Z, then we have, by the principle of conservation of energy,

$$\delta W = -\int (X\delta u + Y\delta v + Z\delta w) d\tau. \tag{13}$$

Comparing the coefficients of δu, δv, δw in (12) and (13), we obtain

$$X = -\Big(\frac{\partial S}{\partial x} \Big)_H + \frac{\partial}{\partial x} \int_0^H k' H dH + \tfrac{1}{2} \frac{\partial}{\partial x} (k''\alpha^2) + \tfrac{1}{2} \frac{\partial}{\partial y} (k''\alpha\beta) + \tfrac{1}{2} \frac{\partial}{\partial z} (k''\gamma\alpha), \tag{14}$$

and analogous expressions for Y and Z.

Assume u, v, w to be infinitely small. Then (14) becomes

$$\left.\begin{aligned} X = -\int_0^H \left(\frac{\partial k}{\partial x}\right)_H H dH + \frac{\partial}{\partial x}\int_0^H k' H dH + \tfrac{1}{2}\frac{\partial}{\partial x}(k'' a^2) \\ + \tfrac{1}{2}\frac{\partial}{\partial y}(k'' a\beta) + \tfrac{1}{2}\frac{\partial}{\partial z}(k'' \gamma a), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (15)$$

These expressions show that we must slightly modify Kirchhoff's expressions for the components of the mechanical force per unit volume, when we consider k and k' to depend upon the strength of the magnetizing force. When we return to the case discussed by Kirchhoff, we see that (15) agrees with his result, excepting differences in notations. The above expressions are not the only forms in which we can write the component forces; they may sometimes be put in the following forms. When $u = v = w = 0$, we have

$$\begin{aligned} \frac{\partial S}{\partial x} &= \left(\frac{\partial S}{\partial x}\right)_H + \frac{1}{4\pi}\left(a\frac{\partial a}{\partial x} + b\frac{\partial \beta}{\partial x} + c\frac{\partial \gamma}{\partial x}\right) \\ &= \left(\frac{\partial S}{\partial x}\right)_H + \frac{1}{4\pi}\left\{\frac{\partial (a a)}{\partial x} + \frac{\partial (b a)}{\partial y} + \frac{\partial (c a)}{\partial z}\right\}, \end{aligned}$$

making use of the equations

$$\frac{\partial \beta}{\partial x} = \frac{\partial a}{\partial y},$$

$$\frac{\partial \gamma}{\partial x} = \frac{\partial a}{\partial z},$$

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0.$$

Consequently (14) becomes

$$\begin{aligned} X &= \frac{\partial}{\partial x}\left\{-S + \frac{a a}{4\pi} + \int_0^H k' H dH + \frac{k'' a^2}{2}\right\} \\ &\quad + \frac{\partial}{\partial y}\left\{\frac{b a}{4\pi} + \frac{k'' a\beta}{2}\right\} + \frac{\partial}{\partial z}\left\{\frac{c a}{4\pi} + \frac{k'' \gamma a}{2}\right\} \qquad (16) \\ &= \frac{\partial}{\partial x}\left\{\left(\frac{1}{4\pi} + k + \frac{k''}{2}\right) a^2 - \frac{H^2}{8\pi} - \int_0^H (k - k') H dH\right\} \end{aligned}$$

$$+\frac{\partial}{\partial y}\left\{\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\alpha\beta\right\}+\frac{\partial}{\partial z}\left\{\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\gamma\alpha\right\}.$$

The corresponding expression for Y and Z can be written down from (16) by symmetry.

The surface-traction $(\overline{X}, \overline{Y}, \overline{Z})$ referred to unit of area can be easily found by integrating the bodily force (X, Y, Z reckoned per unit volume) along the normal to the bounding surface of the solid from one side of the stratum of transition to the other. Let n be the normal to the bounding surface of the solid drawn inwards, and let α', β', γ', H' be the components and the magnitude of the magnetic force in the air. Then in the integral

$$\overline{X}=\int_{(\text{air})}^{(\text{solid})} X dn,$$

we may put

$$X=\cos(n,x)\frac{\partial}{\partial n}\left\{\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\alpha^2-\frac{H^2}{8\pi}-\int_0^H(k-k')HdH\right\}$$
$$+\cos(n,y)\frac{\partial}{\partial n}\left\{\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\alpha\beta\right\}$$
$$+\cos(n,z)\frac{\partial}{\partial n}\left\{\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\gamma\alpha\right\},$$

for the space variation of a quantity in directions perpendicular to n are infinitely small compared with the space variation of the same quantity in the direction of n. Hence

$$\overline{X}=\left(\frac{1}{4\pi}+k+\frac{k''}{2}\right)\alpha\{\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)\}$$
$$-\frac{H^2\cos(n,x)}{8\pi}-\cos(n,x)\int_0^H(k-k')HdH$$
$$-\frac{\alpha'}{4\pi}\{\alpha'\cos(n,x)+\beta'\cos(n,y)+\gamma'\cos(n,z)\}+\frac{H'^2\cos(n,z)}{8\pi}.$$

Since the tangential components of the magnetic force and the normal components of the magnetic induction are continuous at the bounding surface, the above expressions for $\overline{X}$ can be easily put in the form,

$$\overline{X}=-2\pi k^2\{\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)\}^2\cos(n,x)$$

$$- \cos(n, x)\int_0^H (k - k')HdH \tag{17}$$

$$+ \frac{k'a}{2}\{a\cos(n, x) + \beta\cos(n, y) + \gamma\cos(n, z)\}.$$

Here we have, as in the expressions for the components of the bodily force reckoned per unit volume, a term

$$- \cos(n, x)\int_0^H (k - k')HdH$$

involving an integral with respect to H. When k and k' are independent of the strength of the magnetizing force, this term reduces to the form given by Kirchhoff.

We may also find stress components X_x, $X_y \ldots$ $Y_z \ldots$ in ether, and then find the expressions for the components of the surface-traction. We have, assuming the ether to be at rest,

$$\left.\begin{aligned} X &= -\frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z}. \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{18}$$

Comparing (16) and (18) we obtain, as a particular solution of (18),

$$\left.\begin{aligned} X_x &= -\left(\frac{1}{4\pi} + k + \frac{k''}{2}\right)a^2 + \frac{H^2}{8\pi} + \int_0^H (k - k')HdH \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_x = X_z &= -\left(\frac{1}{4\pi} + k + \frac{k''}{2}\right)\gamma a, \\ X_y = Y_x &= -\left(\frac{1}{4\pi} + k + \frac{k''}{2}\right)a\beta. \end{aligned}\right\} \tag{19}$$

Let X_x', X_y', X_z', X_n' be the corresponding quanties in the air. We see at once

$$\begin{aligned} \bar{X} &= X_n' - X_n \\ &= X_x'\cos(n, x) + X_y'\cos(n, y) + X_z'\cos(n, z) \\ &\quad - X_x\cos(n, x) - X_y\cos(n, y) - X_z\cos(n, z). \end{aligned} \tag{20}$$

But

$$X_x' = -\frac{\alpha'^2}{4\pi} + \frac{H'^2}{8\pi},$$

$$X_y' = -\frac{\alpha'\beta'}{4\pi},$$

$$X_z' = -\frac{\gamma'\alpha'}{4\pi},$$

as is easily seen from (19). Hence (20) may be brought to the identical form with (17).

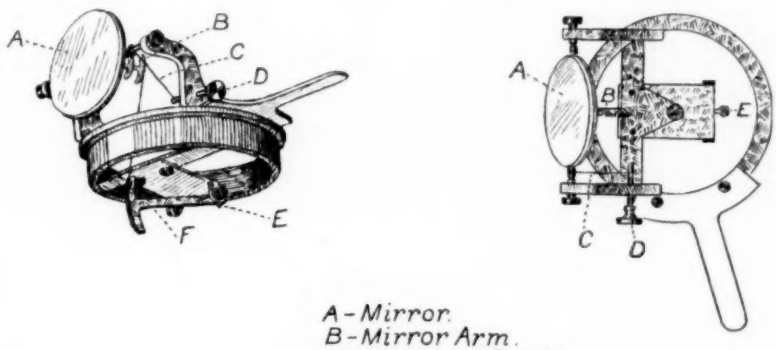
In conclusion I desire to express my best thanks to Prof. A. Tanakadate for his useful suggestions and valuable criticisms. I also wish to thank Mr. K. Honda, who directed my attention to this problem.

PHYSICAL LABORATORY, TOKYO IMPERIAL UNIVERSITY.

THE VOWEL A^E (AS IN HAT).

By Louis Bevier, Jr.

IN a former article (June, 1900) the acoustic character of the vowel *a* (as in *father*) was briefly discussed, as studied from the phonographic record. Since then these researches have been continued on lines similar to those there described. The apparatus employed has been somewhat improved as shown in the accompanying figure. (Fig. 1.) This exhibits the essential parts of the tracer which I have used for the past year. The vibrating mirror has been more carefully mounted than before to secure motion about one axis only. The introduction of an extra lever



A-Mirror.
B-Mirror Arm.
C-Tension Spring.
D-Tension Screw.
E-Sapphire Knob.
F-Lever.

Fig. 1.

provides, on the one hand, for a greater degree of enlargement, and transcribes the elevations and depressions of the phonograph trace without inversion, *i. e.*, right side up. In all other respects the method is the same as in that described in the article to which allusion has been made.

The curves shown in this and the succeeding articles are all made with the same degree of enlargement of ordinates. The lens used

brings the focus to a distance of about 15½ feet from the mirror, giving a beam of light to write with 186 inches long. The arm of the mirror is three-eighths of an inch long. The lever has a short arm of one-eighth of an inch, and a long arm of six-eighths of an inch. Since the long arm of the lever is attached to the arm of the mirror the system gives an enlargement of 5,952 times ($\frac{6}{8} \div \frac{1}{8} = 6$; $186 \div \frac{3}{8} = 496$; $6 \times 496 \times 2 = 5{,}952$). A much greater enlargement may be obtained if desired in two ways, first by using a lens with a longer focal distance, and secondly by reducing the arm of the mirror to two-eighths or to one-eighth of an inch. For practical reasons of convenience this further enlargement has not been resorted to as yet.

The enlargement of abscissas is a matter depending obviously on the speed with which the wax cylinder revolves during the making of a record, and its transcription, and also on the speed with which the drum bearing the photographic paper is driven. Since the present studies concern only vowels at a definite registered pitch, there has been no need to control this matter absolutely, and the speeds used have not been quite the same in all cases. In general, however, the wave as recorded in the wax has been lengthened in transcription about 35 times. In the specimen curves shown in the accompanying plate there may therefore be noted differences in the apparent length of waves of the same pitch. In experiments where the pitch of utterance is not registered it is essential to control carefully the speed of each cylinder.

The present studies have been made from the records of nine different voices, securing enough variety, it is believed, to eliminate from the results conclusions based on peculiarities of utterance that are merely individual. These voices are indicated in the table by the numbers at the extreme left of the page. Numbers 1, 2 and 6 are men's voices of baritone quality; 4 and 5 are children's voices both of soprano quality, the former that of a boy, the latter of a girl; the remaining are women's voices, 8, 9 and 10 sopranos, and 7 a rich contralto.

The waves were measured under a microscope, with appropriate attachments, and were analyzed by the ordinary method from the Fourier series. Generally 36 ordinates were taken, giving the

amplitudes of the fundamental and sixteen upper partials. A few were measured in 24 places and the partials computed as far as XI. In the case of fundamentals of a high pitch, when there was clearly no object in computing the upper partials above V. or VI., two waves were measured as one in 36 places, and the analysis gave the amplitudes of the partials from I. to VIII. These cases may be seen in the following table. It will be observed that the first column contains figures indicating the voice, as above explained. The second column gives the sum of the amplitudes of all the partials, a rough measure of the loudness of utterance. The third column, under I., shows the pitch of the fundamental by means of the vibration number, and its amplitude *percentage*. The fourth column, under II., gives pitch and amplitude percentage of the first upper partial and so on for the rest. In each record the most important partials are indicated by the heavy-faced figures used in expressing their amplitude percentages.

An examination of the above table shows that, no matter what the fundamental or chord-tone may be, the upper partial that is most important by reason of its energy is that whose rate of vibration is in the neighborhood of 1,550 to the second. If it be, as is assumed, an upper partial of the chord-tone, it must of course be a multiple of the vibration-rate of the chord-tone. Save for this relation, it is independent of the pitch of the chord-tone. With a chord-tone of 152 vibrations to the second, it will, in all probability, be the ninth upper partial (X.) which is more prominent, with a vibration rate of 1,520. On the other hand, with a chord-tone of 500 vibrations to the second, the second upper partial (III.) with a vibration rate of 1,500 will be the prominent one. The pitch of the dominant upper partial will be actually a little lower in the second case than in the first. The particular upper partial which is most strongly reinforced by the mouth-resonance will be that which lies nearest to the pitch of maximum resonance of the mouth cavity for the a^e position, *i. e.*, the nearest to 1,550 vibrations to the second. Examples may be taken from the table at will. The dominant upper partial on a fundamental of 202 will be VIII. (1,616), on 226 VII. (1,582), on 256 VI. (1,536), on 320 V. (1,600), and so on. The regularity with which this follows, no matter what the character of the voice examined, is convincing and conclusive.

ANALYSES OF THE A^E CURVE.

		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII
		128	256	384	512	640	768	896	1024	1152	1280	1408	1536	1664	1792	1920	2048	2176
1	118.8	24.9	6.7	2.0	2.0	6.1	**6.6**	5.2	4.2	3.7	0.8	1.9	7.7	**8.8**	8.1	4.9	4.4	2.0
		136	272	408	544	680	816	952	1088	1224	1360	1496	1632	1768	1904	2040	2176	2312
1	54.6	28.6	8.8	1.1	3.7	**4.0**	0.4	5.1	**10.7**	3.7	5.1	7.3	**10.7**	4.0	3.3	1.8	1.4	0.3
		144	288	432	576	720	864	1008	1152	1296	1440	1584	1728	1872	2016	2160	2204	2348
2	64.0	7.8	1.6	4.4	**41.3**	8.4	2.5	**6.6**	3.8	4.7	**9.4**	4.7	1.6	1.3	0.6	0.3	0.3	0.7
		152	304	456	612	760	912	1064	1216	1368	1520	1672						
1	98.6	27.6	2.8	7.4	5.6	**10.6**	5.8	**9.2**	7.0	5.6	**12.6**	5.8						
		160	320	480	640	800	960	1120	1280	1440	1600	1760	1920	2080	2240	2400	2560	2720
1	123.0	9.8	2.3	3.6	**29.9**	2.3	4.9	5.5	8.9	**10.7**	6.2	4.4	1.1	3.3	2.6	1.6	1.8	1.1
6	128.8	45.2	1.8	2.3	**14.7**	1.3	3.5	2.0	3.0	**13.0**	3.6	3.7	1.4	1.2	0.5	1.4	0.4	1.0
		181	362	543	724	905	1086	1267	1448	1629	1810	1991						
1	70.2	20.2	3.7	4.3	**12.5**	9.1	8.0	5.1	8.6	**15.1**	7.0	6.4						
		192	384	576	768	960	1152	1344	1536	1728	1920	2112	2304	2496	2688	2880	3072	3264
1	245.6	14.4	4.1	4.4	**17.8**	2.2	**11.8**	7.0	**11.0**	**11.7**	1.8	5.9	1.6	3.2	0.5	1.3	0.6	0.3
6	77.6	36.5	3.5	**5.3**	**4.9**	4.4	2.4	2.7	**17.8**	8.6	2.6	2.8	2.4	2.2	1.0	1.2	1.2	0.5
7	176.1	62.6	9.7	0.3	**5.0**	2.0	2.6	2.1	**3.2**	**3.4**	1.5	0.9	2.1	1.2	1.5	0.1	1.0	0.8
		202	404	606	808	1010	1212	1414	1616	1818	2020	2222						
2	117.6	6.0	6.0	3.7	**16.5**	6.3	7.1	10.4	**30.1**	1.7	5.5	6.7						
		226	452	678	904	1130	1356	1582	1808	2034	2260	2486						
1	80.8	14.4	3.2	**27.5**	9.9	**10.9**	8.4	**19.5**	3.4	1.0	0.8	1.0						
		256	512	768	1024	1280	1536	1792	2048	2304	2560	2816	3072	3328	3584	3840	4096	4352
2	63.0	14.1	3.8	8.1	**21.5**	10.2	**29.5**	3.2	0.1	2.9	4.1	2.5						
1	444.6	8.4	6.3	17.9	**21.6**	12.1	**11.9**	8.2	3.2	1.3	1.7	0.6	1.0	0.5	0.2	0.3	0.9	1.0
4	177.2	8.6	1.8	3.5	**37.1**	8.5	**10.5**	7.1	8.7	5.0	2.0	1.4	0.8	2.2	0.5	1.0	0.6	0.7

5	142.4	8.4	3.5	7.7	**26.7**	9.6	**19.7**	9.4	3.2	3.1	2.4	1.1	1.7	1.2	0.3	0.7	0.7	0.6
6	163.0	11.0	6.1	**8.1**	3.0	4.2	**28.3**	**17.1**	1.9	1.8	1.5	**3.3**	2.1	2.1	2.4	2.6	3.3	1.2
7	68.2	13.2	4.0	**14.5**	4.0	5.7	**16.9**	**12.0**	5.5	4.8	8.5	4.7	1.9	0.7	1.3	0.9	0.2	1.2
8	122.0	3.5	5.4	**13.9**	**17.5**	6.3	**25.1**	**15.7**	4.6	1.2	0.9	0.5	1.0	1.8	0.9	0.4	0.7	0.6
9	59.6	25.0	2.7	**13.9**	6.4	6.9	**27.0**	7.0	5.2	0.3	1.2	1.5	0.7	1.2	0.2	0.2	0.3	0.3
		272	544	816	1088	1360	1632	1904	2176	2448	2720	2992						
1	182.8	6.0	6.0	**18.6**	13.6	**23.7**	**15.0**	8.7	1.0	2.8	1.9	2.7						
		320	640	960	1280	1600	1920	2240	2560	2880	3200	3520	3840	4160	4480	4800	5120	5440
4	418.2	4.6	6.9	**19.7**	**19.8**	**15.7**	13.2	11.2	2.8	1.2	1.3	0.5	1.5	0.7	0.5	0.2	0.2	0.2
5	207.0	5.5	**17.9**	**25.2**	12.5	**19.5**	9.5	0.6	0.6	2.7	0.4	1.9	0.1	1.8	0.6	0.7	0.2	0.5
7	76.1	3.2	**29.3**	8.4	9.2	**20.1**	6.0	5.2	8.9	3.2	1.2	1.7	0.3	1.2	0.7	0.1	0.3	1.0
8	83.7	6.7	**21.8**	**26.8**	11.5	**20.0**	9.6	0.5	0.1									
9	41.3	6.8	**18.2**	**15.2**	6.5	**28.1**	9.7	3.1	2.4	2.4	1.5	1.7	0.7	0.5	0.5	0.2	1.3	1.2
10	84.3	6.5	**13.1**	**21.7**	**19.6**	**21.5**	9.4	1.4	1.8	0.6	1.7	0.5	0.5	0.1	0.6	0.2	0.1	0.7
		384	768	1152	1536	1920	2304	2688	3072	3456	3840	4224	4608	4992	5378	5760	6144	6528
4	250.6	2.4	7.5	**51.9**	**10.5**	9.0	9.5	1.4	0.3	1.6	1.0	1.2	0.7	0.5	0.3	0.8	0.7	0.7
5	285.0	9.3	14.2	**31.9**	**25.8**	9.4	3.5	3.6	2.3									
7	302.7	6.6	**23.4**	8.3	**42.8**	12.8	4.3	0.7	1.1									
8	108.0	4.9	13.7	**28.5**	**38.7**	8.1	0.5	2.8	2.8									
9	128.5	7.8	13.7	**17.4**	**42.3**	13.6	1.1	0.2	0.5	0.8	0.2	0.1	0.5	0.2	0.6	0.5	0.2	0.1
		480	960	1440	1920	2400	2880	3360	3840									
4	220.3	9.9	19.9	**33.3**	**20.5**	4.9	5.5	4.0	2.0									
		512	1024	1536	2048	2560	3072	3584	4096									
4	130.9	8.2	**25.4**	**30.3**	20.0	9.9	2.8	1.8	1.6									
5	271.0	6.4	**52.0**	**29.4**	4.8	3.6	2.6	0.6	0.6									
7	378.1	7.5	**24.5**	**54.9**	5.1	2.1	4.6	0.7	0.6									
8	270.7	6.3	**52.3**	**26.8**	6.0	4.2	2.7	1.1	0.6									
9	150.4	4.9	**44.8**	**39.9**	4.0	3.1	0.7	2.3	0.3									
		640	1280	1920	2560	3200	3840	4480	5120									
10	95.1	46.3	**29.4**	**16.6**	3.3	1.8	0.5	1.0	1.1									

It must also be noted that there are often two or even three consecutive partials that appear with considerable amplitude, *i. e.*, that are reinforced in the mouth cavity. This occurs particularly when the fundamental or chord-tone is low, and again when it is above 300. The explanation of the second case will appear later. Here let us observe the facts of the first case. When the chord-tone is of low pitch, *e. g.*, under 150 to the second, the upper partials lie of course near together. Take, for example, the fundamental 128 in the table. Here XII., XIII., and XIV. are all reinforced. In this particular utterance the pitch of maximum resonance was a little higher than an ordinary a^e, and XIII. with a vibration rate of 1,664 is most strongly reinforced, but XII. and XIV. (1,536 and 1,792 respectively) are also strongly reinforced. This is most important for the theory of the vowels, for it shows in a clear way what might well have been anticipated theoretically, viz., that the mouth cavity, with its irregular shape and walls of varying degrees of elasticity and softness, has no one particular and exact proper tone (*Eigenton*), but rather resounds to any pitch within certain not very narrow limits around a maximum. We have then a region of resonance with a not very pronounced maximum point, instead of a particular and exact pitch. For a^e this region of resonance centers at about 1,550 but often extends from 1,400 to 1,700 or even higher. Plainly then with a low fundamental we are to expect the reinforcement of adjacent upper partials.

In addition to this most characteristic resonance region, as just explained, the vowel a^e shows generally two other lower resonances. These are not so uniform and hence not so characteristic, but one or both of them *must* be present to give the a^e color. These lower regions of resonance center respectively at 1,050 and 650. For convenience I have indicated by full-faced type the important upper partials, both those due to the lower resonances, and to the upper or more characteristic one. For detailed examination take the record of voice I., on the fundamental 226. Here III. falls at 678, very near a maximum of resonance, and has in consequence a large amplitude percentage (27.5). On the other hand, neither IV. (904), nor V. (1,130) lies very near the next maximum at 1,050, and while both are reinforced, and are present with note-

worthy percentages (9.9 and 10.9 respectively), neither is relatively very important. Again VI. (1,356) is not high enough to fall within the upper region of full resonance, and is not powerfully reinforced, but VII. (1,582) coincides almost exactly with the upper maximum, and has a large amplitude percentage (19.5). Its energy percentage would of course be far greater. Many records show distinctly the three resonance regions which I have noted, as for example 1 (192) and 1 (226) in the table.

On the other hand many records, on analysis, are found to lack one or the other of these lower resonances. Both examples at 160 show IV. strong at 640, but scarcely any indication of any resonance at or near 1,050, whereas the first example at 320 shows weak resonance at 640. The example at 202 is unusual, showing a strong IV. and a weak III. and V., whereas we should normally expect a weak IV. and a strong III. and V. Such individual differences are not surprising, and our definition of any vowel must be broad enough to cover them.

We are now able to understand why it is that on high fundamentals or chord-tones we often find consecutive upper-partials strongly reinforced. It may well happen that one upper-partial will fall within one of the regions of resonance, and the next within another region of resonance, when the pitch interval between one upper-partial and the next is a wide one. At 512, for example, II. (1,024) and III. (1,536) should both of them be strongly reinforced, and at 384 it may well be that II. (768), III. (1,152) and IV. (1,536) are all strong. It must also be observed that in general the amplitude percentages are higher on a high fundamental, because there are fewer upper partials available to take the entire resonance energy. It is in accord, too, with the whole theory of resonance that the upper partials near the fundamental are favored, other things being equal. Hence if the mouth cavity resounds weakly to the vibration rate 900, on a fundamental of 150, we shall be sure to find VI. very weak, but on a fundamental of 450 II. will be fairly strong. It often happens that, where a fundamental is of high pitch, no one of the upper partials coincides well with any region of strong resonance. We have then forced resonance, the resonator adjusting itself as best it can. Thus it comes to pass that as the pitch of the

chord-tone is raised, after the tones of ordinary speech are passed, the differentiation between the vowels becomes less and less perfect, until at 1,024 (soprano high c) little differentiation is possible. This difficulty does not arise, however, at any pitch used in the ordinary speech either of men's or of women's voices.

As a result of these observations I would define our American *a*^{e} as consisting of the following acoustic elements:

(1) A chord-tone more or less strong according to the amount of reinforcement which the mouth cavity gives it, varying within great range. In general it is quite strong below 200, and weak between 250 and 600, above which last point it becomes very strong, coinciding with the lowest observed mouth-resonance of this vowel.

(2) A characteristic *a*^{e} resonance in the region of 1,550, *always* present, but not absolutely stationary, because the vowel may tend more or less toward *a* or toward *e*, with endless nuances. This characteristic resonance is not a particular point, but covers a consider able region.

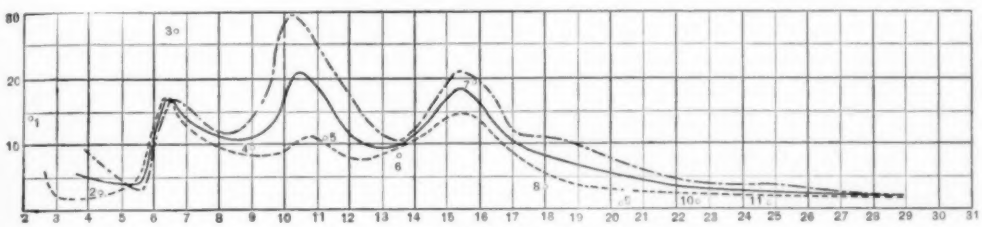
(3) A strong resonance at one or two lower regions centering in 650 and 1,050 respectively. Both *may* be present, but one at least *must* be. Great individual variation is shown in the relation of these resonances to each other and to the fundamental.

The characteristic *a*^{e} resonance at (*ca*) 1,550 largely determines the vowel character, but it must receive "body," so to speak, by one or more reinforced lower tones. This gives it to the ear a certain fullness or openness which it requires, setting it in sharp contrast to the thinness of *i* (as in *pique*), whose characteristics will be discussed later.

These facts may be exhibited diagrammatically by plotting a resonance curve for *a*^{e}. This I have done by taking all my *a*^{e} records, at least twice as many as are shown in the table of analyses, and averaging their resonance percentages. The characteristic resonance-curve of *a*^{e}, thus obtained, is shown in Fig. 2. The upper line of dots and dashes is the average of the records of women's and children's *a*^{e}'s, the lower broken line that of men. The continuous line shows the average for all taken together. This curve shows the mouth-resonance rising sharply from 550 and reaching a maximum at 650. From this point it declines, rising to a second max-

imum at 1,050. From here it once more declines and rises to its upper and most characteristic maximum at 1,550. The women's line is higher than the men's for reasons already sufficiently explained.

This resonance-curve is only an average, and as in the case of all averages is not altogether true. It represents the vowel a^e much as a composite photograph represents a human face. No individual utterance is likely to coincide closely with it in every part. To illustrate this I have introduced for comparison an individual record made on a fundamental or chord-tone of 226. The amplitude of each partial is plotted, $\odot^1$ being the fundamental, and $\odot^2$, $\odot^3$, $\odot^4$, etc., the upper partials. It will be seen that II. is weak as it should be, III. is very strong, much above the average, IV., V., VI. follow



A^E RESONANCE CURVE

the men's line closely, VII. is at a maximum, VIII. is low when it should be, and IX., X. and XI. are negligible. If the curve were plotted as an energy curve (cf. PHYS. REV., X., 4, p. 200), the dominating importance of the highest resonance would be more prominent to the eye.

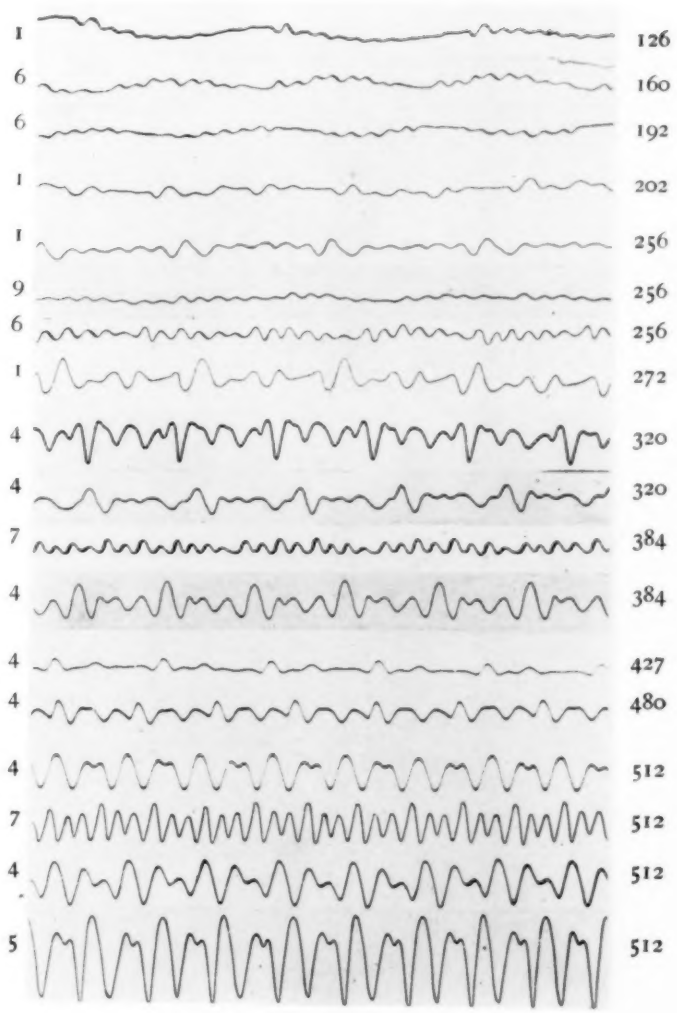
It is instructive to study the actual curves obtained. The accompanying plate shows a sheet of a^e curves reduced to about ⅓ actual size. The upper resonance is in general easy enough to detect by the simple process of counting the small waves embroidered upon the fundamental wave. For example at 256 in each of the specimens shown there are six crests plainly marked on each of the long or fundamental waves, and 6×256 gives the characteristic maximum of 1,536. At 512 the crests are three in number, etc. As a rough and ready means of discrimination the eye is usually quite sufficient if the pitch of the fundamental is known. For the

lower resonances the eye may still suffice sometimes after practice, *e. g.*, at 512, where it is evident at a glance (if it be a practiced glance) that II. is present in large amplitude. But for the accurate determination of relative amplitudes the mathematical analysis is indispensable. The correct analysis of a complex curve, such as either of the examples at 320, is entirely beyond the eye alone. At the lower pitches the interference of adjacent upper partials is often quite apparent to the eye but the determination of numerical relations cannot be made by the unaided eye.

It is apparent from the above why all *a*'s, whether pronounced by man, woman, or child, are heard by the ear as a unit sound. There is a large element which is substantially a constant. The variable chord-tone does not disguise the identity, for the ear has learned by long practice to disassociate this from the function of vowel-differentiation. On the other hand it is equally clear that no one *a* is ever quite like any other, and that, without violating the essential characteristics of the vowel there is room for infinite variety of individual utterance. We may learn to recognize a friend's voice in his pronunciation of a single vowel sound. Not only so, but in the whispered voice, where the chord-tone is absent, and the mouth resonances alone are present, weakly called forth by the force of expiration, and accompanied by many slight noises of rushing breath, the same essential identity is clearly possible, with the same infinite variety of individual utterance.

RUTGERS COLLEGE,
NEW BRUNSWICK, N. J.

(To face p. 180.)



A°—Curves

MAGNETIZATION OF STEEL AT LIQUID AIR TEMPERATURES.

By C. C. Trowbridge.

THE present paper is an account of experiments on the magnetic moments of small bars of steel magnetized at the temperature of boiling liquid air.

The magnetic moments obtained by magnetization at approximately — 185° C. were compared with the magnetic moments of bars of the same steel obtained by magnetization at normal temperature, 20° C.

The change of magnetic strength produced by heating bar magnets to 20° C. which had been magnetized at — 185° C. was also determined, and compared with the change produced by cooling magnets of the same steel to — 185° C. which had been magnetized at 20° C. The experiments were chiefly confined to carbon magnet steel and tungsten magnet steel.

Dewar has investigated the effect of lowering the temperature of permanent magnets to — 182° C. and determining the magnetic moments at + 15° and — 182° C. through several cycles of temperature change between these limits.[1] Later, more extensive experiments of the same nature were performed by Dewar and Fleming together.[2] Among the conclusions reached by them are the following:

"1. That the sudden cooling to the temperature of liquid air usually permanently decreases the magnetic moment of short magnets made of many varieties of steel, assuming them to have been initially magnetized in a strong field.

"2. This initial decrease is found both in hardened steels having great coercive force, and also in the same steels in a soft or an-

[1] Chemical News, April 26, 1895.

[2] Proceedings Royal Society, Vol. 60, p. 57, 1896.

nealed condition and especially conspicuous in the case of the 19 per cent. nickel steel.

"3. In the case of most steels so far examined the effect of cooling magnets made of them to — 185° C. is to temporarily increase the magnetic moment after the permanent magnetic condition has been reached.

"4. The exceptions of the above rule so far noted, are the nickel steels with percentages of nickel from 19 to 29 per cent., in which case the magnetic moment is always decreased temporarily by cooling to — 185° C. after the permanent magnetic condition has been reached.

In the experiments just referred to, performed by Dewar and Fleming, the steels tested were magnetized at normal temperatures and then subjected to temperature changes through the range + 15° to — 185° C.

In the author's experiments described below, steel bars were magnetized at the temperature of boiling liquid air and their magnetic moments determined at the same temperature.

The method employed was as follows:

Short bars of steel of equal length, having been cut from the same long bar, were heated together to a red heat and then plunged into water at 20° C.; they were thus hardened to approximately the same degree.

In order to magnetize the steel bars at — 185° C., a small silvered cylindrical Dewar bulb of 100 cc. capacity, was filled with liquid air and placed within a coil having 6,500 turns of copper wire. The bar of steel to be magnetized was then put into the Dewar bulb and kept in the center of the coil by two perforated cardboard disks which fitted the interior of the bulb.

The arrangement is shown in Fig. 1, in which letters indicate the parts of the apparatus as follows: *C*, coil; *D*, Dewar bulb; *A*, liquid air; *K*, cardboard disks; *M*, magnet.

The dimensions of the coil employed were as follows:

Length, 9 cm.; inside diameter, 8 cm.; outside diameter, 14.5 cm.

The coil was made of 6,500 turns of No. 24 B. & S. copper wire, and had a resistance of 203 ohms.

The magnetizing force was very strong, .29 ampère being used; this produced 1,885 ampère turns.

The time of magnetization was made 60 seconds in each case.

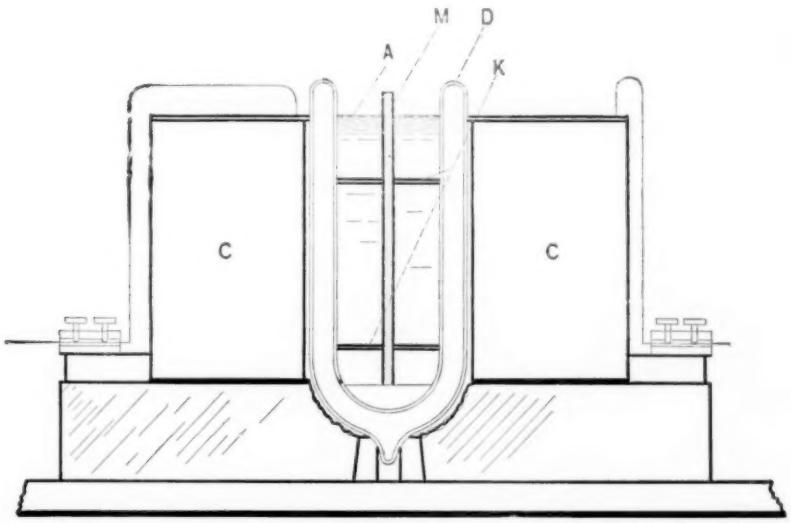


Fig. 1.

After the bars of steel were magnetized at $-185°$ C. they were kept immersed in liquid air until the magnetic moment at the low temperature was determined.

The magnetic moments of the magnets were determined with an ordinary magnetometer, the magnets resting in a brass tray, the dimensions of which were $10 \times 1.5 \times 1.5$ cm.

The method is indicated by Fig. 2, in which T is a brass tray; M, a magnet; H, a magnetometer, W; a wooden carrier for the tray; S, a meter scale.

When the magnetic moments at low temperature were determined the brass tray was filled with liquid air.

The initial change of magnetic strength, or the first gain or loss in the magnetic moment, accompanying the first change of temperature of the bars of steel after they were magnetized was determined, because it has been shown by Dewar and others that the change of magnetic strength of a permanent magnet due to a change of temperature depends partly on the previous history of the magnet.

Special care was taken not to jar the magnets throughout the experiments, and also to determine their magnetic moments immediately after magnetization; to these precautions is probably due the good agreement of the results.

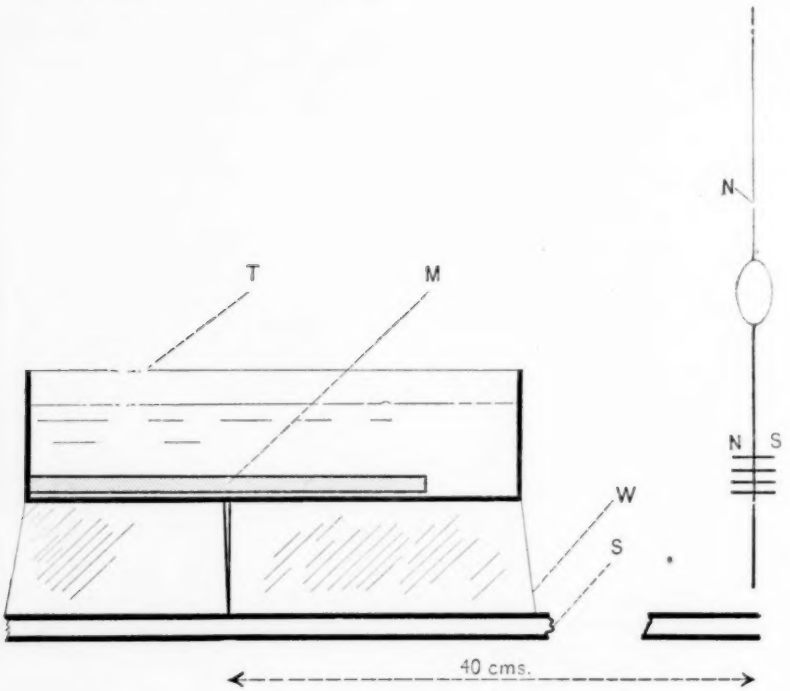


Fig. 2.

The experiments were interrupted by the failure of the company from which liquid air was produced, but the results already obtained show some interesting facts.

Two kinds of steel specially made for use as permanent magnets were tested. These steels were:

Carbon Steel, Crescent Company, Pittsburg, containing 2.40 per cent. of carbon.

Tungsten steel from Sheffield, England. (Percentage of tungsten not given.) The results of the experiments are given below in tabular form:

Carbon Steel, Crescent Company, Pittsburg — 2.4 per cent carbon.

Three bars of carbon steel, size, 83 x 10 x 3 millimeters, weighing 24.45 grams, were heated together to a red heat and then hardened in water at 20° C. They were magnetized at −185° C. and then immediately warmed to 20° C. The magnetic moments at these temperatures were as follows:

TABLE I.

Carbon Steel Magnetized at −185° C.

Magnet.	Magnetic Moment at −185° C.	Magnetic Moment at 20° C.	Difference (loss).	Percentage Loss.
1	211	131	80	37.9
2	202	140	62	30.6
3	211	147	64	30.3
Mean				32.9

Another bar of this steel of the same dimensions and weight as the magnets 1, 2 and 3 and hardened with them, was magnetized at 20° C. and then immediately cooled to −185° C.

The magnetic moments at 20° C. and −185° C. were as follows:

TABLE II.

Carbon Steel, Magnetized at 20° C.

Magnet.	Magnetic Moment at 20° C.	Magnetic Moment at −185° C.	Difference (Loss).	Percentage Loss.
4	208	189	19	9.1

It is evident that the initial loss in magnetic moment due to change of temperature when the steel is magnetized at −185° C., and then warmed to 20° C., is much greater than for the reverse operation, although in either case, a considerable loss in the magnetic moment is the result of a change in temperature of the steel of 205°.

These carbon steel magnets, 1, 2, 3 and 4, were allowed to rest undisturbed for nine days at approximately constant temperature, (room temperature in April). Their magnetic moments were then redetermined and are given in Table III, page 186.

TABLE III.

Carbon Steel; Total Change of Magnetic Moment at the end of Nine Days.

Magnet.	Original Magnetic Moment.	Magnetic Moment after 9 Days.	Difference (Loss).	Percentage Loss.
1	211	127	84	39.8
2	202	136	66	32.6
3	211	138	73	34.6
4	208	170	38	18.2

Magnets 1, 2 and 3 were magnetized at —185° C. and then warmed to 20° C. The total loss of magnetic strength of these magnets since magnetization was approximately twice as much as magnet 4, which was magnetized at 20° C., cooled to —185° C. (with a loss of 9.1 per cent), and then warmed to 20° C. again.

Tungsten Steel from Sheffield.—Three bars of tungsten steel, size 89 x 2 x 3 millimeters, each weighing 5.05 grams, were hardened together in water at 20° C. They were magnetized at — 185° C., and then warmed to 20° C.

The magnetic moments at the two temperatures were as follows:

TABLE IV.

Tungsten Steel, Magnetized at —185° *C.*

Magnet.	Magnetic Moment at —185° C.	Magnetic Moment at 20° C.	Difference (Loss).	Percentage Loss.
1	212	179	33	15.5
2	228	192	36	15.7
3	232	206	26	11.2
Mean				14.1

Magnet 1, nine days after showed a magnetic moment of 176.

Magnet 2 was cooled to — 185° C. immediately after the magnetic moment 192 was obtained and showed a moment of 196, a gain of about 2 per cent. of the value 192.

The low percentage loss (11.2), of magnet 3 may have been due to a slight mistake in manipulation that was made, which may have introduced an error of several per cent. This determination (magnet 3) is therefore questionable to that extent.

A bar of this steel hardened with magnets 1, 2, and 3 was magnetized at 20° C., and then cooled to — 185° C. The magnetic moments obtained were as follows:

TABLE V.

Tungsten Steel, Magnetized at 20° C.

Magnet.	Magnetic Moment at 20° C.	Magnetic Moment at —185° C.	Difference (Loss).	Percentage Loss.
4	213	200	13	6.1

Bundle of iron wires (not pure iron), 9 cm. long heated to a red heat and cooled in air at 20° C., magnetized at —185° C., then warmed to 20° C., showed the following loss of magnetic strength.

TABLE VI.

Iron Magnetized at —185° C.

Magnetic Moment at —185° C.	Magnetic Moment at 20° C.	Difference (Loss).	Percentage Loss.
22.3	13.5	8.8	39.4

Another fact is shown by these experiments; approximately the same magnet moment is produced, whether the steel is magnetized at 20° C., or —185° C., other conditions being equal. This is evident from a comparison of the magnetic moments obtained after magnetizing steel bars at —185° C., with the magnetic moment obtained after magnetizing a bar of the same steel at 20° C.

There is a slight tendency towards a higher magnetic moment at low temperature than at normal temperature, but no definite statement can be made in regard to this point with the few determinations made.

TABLE VII.

Carbon Steel, Magnetized at —185° C. and at 20° C.

Magnet	Magnetized at.	Magnetic Moment.
1	—185° C.	211
2	—185° C.	202
3	—185° C.	211
4	+ 20° C.	208

TABLE VIII.

Tungsten Steel, Magnetized at −185° *C. and at* 20° *C.*

Magnet.	Magnetized at	Magnetic Moment.
1	−185° C.	212
2	−185° C.	228
3	−185° C.	232
4	+ 20° C.	213
5	−185° C. + 20 C.	232

Three bars of tungsten steel were magnetized at −185° C., but the time of magnetization was made different in each case. A bar of the same steel hardened with these magnets was magnetized at 20° C. for comparison. The magnetic moments obtained are shown in Table IX.:

TABLE IX.

Magnet.	Magnetized at	Magnetic Moment.
9	−185° C. for 1 sec.	136
10	−185° C. " 6 "	145
11	−185° C. " 60 "	145
12	20° C. " 60 "	145

These experiments were for the purpose of determining if there was any decided lag in the production of magnetic saturation at low temperature, and were mainly of a precautionary nature. It is evident that an approximate state of saturation is reached far inside the time of magnetization allowed (sixty seconds), in the tests of tungsten and carbon steels.

In the last experiments with tungsten steel the determinations were relative only, and were made with a magnetometer of special design. The instrument, a "differential astatic magnetometer," was suggested by Professor Rood and constructed by the author.

The essential part of the instrument is the suspension system, which consists of two groups of small magnets, set 23 cm. apart, rigidly connected by a very fine glass rod. The system is suspended by a single raw silk fiber 10 cm. long. By making the polarity of the two groups of magnets opposite, a system that is approximately astatic is obtained.

The object of the arrangement employed is partly to annul the effects of distant magnetic disturbing influences, such as those that arise from trolley car motors, etc., and partly to obtain a sensitive system that will act on the differential principle.

A magnet placed within a meter of the instrument and outside of the neutral plane between the two groups of magnets acts strongly on the nearest group, producing a deflection of the system.

In the foregoing experiments on the magnetization of steel at liquid air temperature, it was desirable to eliminate the magnetic effect of the liquid oxygen in the liquid air in the Dewar bulb, during the process of magnetization. It was considerd advisable to use a very strong magnetizing force. This produced approximate saturation at both low and normal temperatures, rendering the magnetic effect of the liquid oxygen on the magnetic moments obtained by magnetization at low temperature practically negligible.

The permeability of liquid oxygen is comparatively low, moreover, as can be seen from the results of an investigation of Fleming and Dewar,[1] who have studied the magnetic permeability of liquid oxygen and liquid air. They find that the magnetic permeability of liquid oxygen is 1.00287. This number is the ratio of the magnetic permeability of liquid oxygen to that of cold gaseous oxygen, and by the same method used in the given determination no difference was detected between the magnetic permeability of gaseous oxygen at — 182° C. and at normal temperatures. The value 1.0024 was obtained for air.

These values are far below that of the so-called non-magnetic iron alloys. Twelve per cent. manganese steel, usually called non-magnetic, has a magnetic permeability of 1.3 or 1.4.

As previously stated, the magnetic moments of the steel bars were determined with an ordinary magnetometer, but the magnets while being tested at low temperatures were placed in a small tray filled with liquid air. The magnetic effect of the liquid oxygen in the liquid air used in this part of the experiments was considered negligible also, for a tray containing liquid air alone was placed with several centimeters of the magnetometer and produced practically no effect on it. When the magnets were tested, they were

[1] Proc. Royal Society, 1896, Vol. 60, p. 283.

placed at 40 cms. from the magnetometer. Moreover, the liquid air in the tray being allowed to boil away, no appreciable change was noticed in the deflection of the magnetometer immediately afterwards.

It is appropriate here to call attention to the fact that in several cases mistakes have been made by writers in stating the results of investigations on magnetism at low temperature. Among these is the following statement taken from Thompson's "Elementary Lessons in Electricity and Magnetism," edition of 1901, p. 104.

"Trowbridge found severe cooling at 100° below zero to destroy the magnetism of steel magnets; but Dewar has observed that when cooled to — 200° C. in liquid oxygen, the magnetic properties of iron are nearly twice as high as at 0° C."

What Professor John Trowbridge really found was as follows:

That a bar magnet magnetized at 6° or 8° C. when cooled from 4° C. to — 25° lost less than 4 per cent. of magnetism. For a greater reduction of temperature to about — 60° C. a far greater percentage loss of magnetism was observed. In one case, a bar magnet magnetized to saturation lost 66 per cent. of its magnetism.[1]

The latter part of the statement quoted above probably refers to an investigation of Fleming and Dewar,[2] in which they found that the magnetic permeability of soft annealed Swedish iron decreased when cooled to low temperatures, but that the permeability of unannealed Swedish iron increased as the temperature diminished.

In the case of hardened iron the permeability increased several hundred per cent. when the iron was cooled to liquid air temperature.

Permanent magnets made from commercial steels containing various percentages of tungsten, etc., uncertain quantities of carbon, and having different degrees of hardness, must be regarded, in general, as individuals with special characteristics; yet the following conclusions can be drawn from the experiments described above:

1. That approximately the same magnetic moment is obtained whether a bar of tungsten steel or carbon steel is magnetized at normal or at liquid air temperature, other conditions being the same.

[1] American Journal of Science, Vol. XXI., p. 316, 1881.

[2] Proceed. Royal Soc., Vol. 60, p. 81, 1896.

2. That the initial loss in the magnetic moment, due to change of temperature, of a bar of steel magnetized at $-185°$ C. and then heated to 20° C. is much greater than when the bar is magnetized at 20° C. and then cooled to $-185°$ C., a considerable loss occurring in both cases.

3. That there is a certain amount of unstable magnetism in a newly made steel magnet, which tends to pass off at the first change of temperature in either direction from that at which magnetization takes place, much more of the unstable magnetism passing off by heating than by cooling.

It was determined that the tungsten steel magnets, while nearly the same length, but only about one-fifth of the mass of the carbon steel magnets, give a higher magnetic moment than the carbon steel. Some allowance must be made, however, for the fact that the latter, on account of their greater mass, were probably not hardened to the same degree as the tungsten magnets, the two steels being hardened in the same manner. Furthermore, the initial losses of magnetic strength due to change of temperature from $-185°$ C. to 20° C. and from 20° C. to $-185°$ C. were very much less in the case of tungsten steel magnets than for the carbon steel magnets. It is evident that there is less of the unstable magnetism, referred to above, in the case of magnets made of tungsten steel than for those made of carbon steel.

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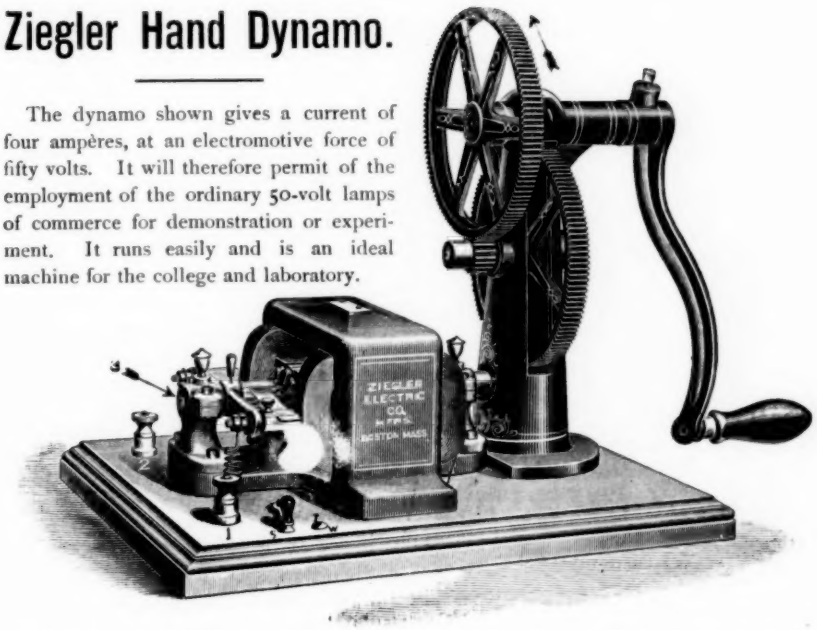

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